

# AUTOMORPHISMS OF LOCAL FIELDS OF PERIOD $p^M$ AND NILPOTENT CLASS $< p$

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**ABSTRACT.** Suppose  $K$  is a finite extension of  $\mathbb{Q}_p$  containing a  $p^M$ -th primitive root of unity. For  $1 \leq s < p$  denote by  $K[s, M]$  the maximal  $p$ -extension of  $K$  with the Galois group of period  $p^M$  and nilpotent class  $s$ . We apply the nilpotent Artin-Schreier theory together with the theory of the field-of-norms functor to give an explicit description of the Galois groups  $\text{Gal}(K[s, M]/K)$ . As application we prove that the ramification subgroup  $\Gamma_K^{(v)}$  of the absolute Galois group of  $K$  acts trivially on  $K[s, M]$  if and only if  $v > e_K(M + s/(p-1)) - (1 - \delta_{1s})/p$ , where  $e_K$  is the ramification index of  $K$  and  $\delta_{1s}$  is the Kronecker symbol.

## INTRODUCTION

Everywhere in the paper  $M \in \mathbb{N}$  is fixed and  $p \neq 2$  is prime.

Let  $K$  be a complete discrete valuation field of characteristic 0 with finite residue field  $k \simeq \mathbb{F}_{q_0}$ , where  $q_0 = p^{N_0}$ ,  $N_0 \in \mathbb{N}$ . Fix an algebraic closure  $\bar{K}$  of  $K$  and denote by  $K_{<p}(M)$  the maximal  $p$ -extension of  $K$  in  $\bar{K}$  with the Galois group of nilpotent class  $< p$  and exponent  $p^M$ . Then  $\Gamma_{<p}(M) := \text{Gal}(K_{<p}(M)/K) = \Gamma/\Gamma^{p^M}C_p(\Gamma)$ , where  $\Gamma = \text{Gal}(\bar{K}/K)$  and  $C_p(\Gamma)$  is the closure of the subgroup of commutators of order  $\geq p$ .

Let  $\{\Gamma^{(v)}\}_{v \geq 0}$  be the ramification filtration of  $\Gamma$  in upper numbering [14]. The importance of this additional structure on the Galois group  $\Gamma$  (which reflects arithmetic properties of  $K$ ) can be illustrated by the local analogue of the Grothendieck Conjecture [13, 4, 5]: the knowledge of  $\Gamma$  together with the filtration  $\{\Gamma^{(v)}\}_{v \geq 0}$  is sufficient to recover uniquely the isomorphic class of  $K$  in the category of complete discrete valuation fields.

Let  $\{\Gamma_{<p}(M)^{(v)}\}_{v \geq 0}$  be the induced ramification filtration of  $\Gamma_{<p}(M)$ . Then the problem of arithmetical description of  $\Gamma_{<p}(M)$  is the problem of explicit description of the filtration  $\{\Gamma_{<p}(M)^{(v)}\}_{v \geq 0}$  in terms of generators of  $\Gamma_{<p}(M)$ .

An analogue of this problem was studied in [1, 2, 3] in the case of local fields  $\mathcal{K}$  of characteristic  $p$  with residue field  $k$ . More precisely, let  $\mathcal{G} = \text{Gal}(\mathcal{K}_{sep}/\mathcal{K})$  and  $\mathcal{G}_{<p}(M) = \mathcal{G}/\mathcal{G}^{p^M}C_p(\mathcal{G})$ . In [1, 2] we developed a nilpotent version of the Artin-Schreier theory which allows us to construct identification of profinite groups  $\mathcal{G}_{<p}(M) = G(\mathcal{L})$ . Here  $\mathcal{L}$

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is a profinite Lie  $\mathbb{Z}/p^M$ -algebra of nilpotent class  $< p$  and  $G(\mathcal{L})$  is the pro- $p$ -group, obtained from  $\mathcal{L}$  by the Campbell-Hausdorff composition law, cf. Subsection 1.2 below for more details and Subsection 1.1 in [7] for non-formal comments about nilpotent Artin-Schreier theory.

On the one hand, the above identification of  $\mathcal{G}_{<p}(M)$  with  $G(\mathcal{L})$  depends on a choice of uniformising element in  $\mathcal{K}$  and, therefore, is not functorial (in particular, it can't be used directly to develop a nilpotent analog of classical local class field theory). On the other hand, the ramification subgroups  $\mathcal{G}_{<p}(M)^{(v)}$  can be now described in terms of appropriate ideals  $\mathcal{L}^{(v)}$  of the Lie algebra  $\mathcal{L}$ . The definition of these ideals essentially uses the extension of scalars  $\mathcal{L}_k := \mathcal{L} \otimes W_M(k)$  of  $\mathcal{L}$  (such operation does not exist in the category of  $p$ -groups) together with the appropriate explicit system of generators of  $\mathcal{L}_k$ , cf. Subsection 1.4. This justifies the advantage of the language of Lie algebras in the theory of  $p$ -extensions of local fields.

In this paper we apply the above characteristic  $p$  results to the study of similar properties in the mixed characteristic case, i.e. to the study of the group  $\Gamma_{<p}(M)$  together with its ramification filtration. Our main tool is the Fontaine-Wintenberger theory of the field-of-norms functor [15]. Note also that we assume that  $K$  contains a primitive  $p^M$ -th root of unity and our methods generalize the approach from [8] where we considered the case  $M = 1$ . In some sense our theory can be treated as nilpotent version of Kummer's theory in the context of complete discrete valuation fields. As a result, we identify  $\Gamma_{<p}(M)$  with the group  $G(L)$ , where  $L$  is a Lie  $\mathbb{Z}/p^M$ -algebra and for an appropriate ideal  $\mathcal{J}$  of  $\mathcal{L}$ , we have the following exact sequence of Lie algebras

$$(0.1) \quad 0 \longrightarrow \mathcal{L}/\mathcal{J} \longrightarrow L \longrightarrow C_M \longrightarrow 0.$$

Here  $C_M$  is a cyclic group of order  $p^M$  with the trivial structure of Lie algebra over  $\mathbb{Z}/p^M$ .

As a first step in the study of  $L$ , we give an explicit description of the ideal  $\mathcal{J}$ . More generally, if  $C_s(L)$  is the closure of the ideal of commutators of order  $\geq s$  in  $L$ , then for  $s \geq 2$ , we have  $C_s(L) \subset \mathcal{L}/\mathcal{J}$  and exact sequence (0.1) induces the exact sequences

$$0 \longrightarrow \mathcal{L}/\mathcal{L}(s) \longrightarrow L/C_s(L) \longrightarrow C_M \longrightarrow 0,$$

where all  $\mathcal{L}(s)$  are ideals in  $\mathcal{L}$ . The main result of Section 3, Theorem 3.3, describes these ideals  $\mathcal{L}(s)$  with  $2 \leq s \leq p$  and gives in particular that  $\mathcal{J} = \mathcal{L}(p)$ .

Extension (0.1) splits in the category of  $\mathbb{Z}/p^M$ -modules and its structure can be given by explicit construction of a lift  $\tau_{<p}$  of a generator of  $C_M$  to  $L$  and the appropriate differentiation  $\text{ad}\tau_{<p} \in \text{End}(\mathcal{L}/\mathcal{J})$ . The study of  $\text{ad}\tau_{<p}$  will be done in the next paper via methods used in the case  $M = 1$  in [8].

In Section 4 we apply our approach to find for  $1 \leq s < p$ , the maximal upper ramification numbers  $v(K[s, M]/K)$  of the maximal extensions  $K[s, M]$  of  $K$  with Galois groups of period  $p^M$  and nilpotent class  $s$ . (The maximal upper ramification number for a finite extension  $K'/K$  in  $\bar{K}$  is the maximal  $v_0$  such that the ramification subgroups  $\Gamma^{(v)}$  act trivially on  $K'$  if  $v > v_0$ .) This result can be stated in the following form, cf. Theorem 4.5 from Section 4:

- If  $[K : \mathbb{Q}_p] < \infty$  and  $\zeta_M \in K$  then for  $1 \leq s < p$ ,

$$v(K[s, M]/K) = e_K \left( M + \frac{s}{p-1} \right) - \frac{1 - \delta_{s1}}{p}.$$

where  $e_K$  is the ramification index of  $K/\mathbb{Q}_p$  and  $\delta$  is the Kronecker symbol.

**Remark.** The case  $s = 1$  is very well-known and can be established without the assumption  $\zeta_M \in K$ . Is it possible to remove this restriction when  $s > 1$ ?

**Notation.** If  $\mathfrak{M}$  is an  $R$ -module then its extension of scalars  $\mathfrak{M} \otimes_R S$  will be very often denoted by  $\mathfrak{M}_S$ , cf. also another agreement in Subsection 1.1. Very often we drop off the indication to  $M$  from our notation and use just  $K_{<p}, \Gamma_{<p}, \mathcal{G}_{<p}$  etc. instead of  $K_{<p}(M), \Gamma_{<p}(M), \mathcal{G}_{<p}(M)$ , etc.

## 1. PRELIMINARIES

Let  $\mathcal{K}$  be a complete discrete valuation field of characteristic  $p$  with residue field  $k \simeq \mathbb{F}_{q_0}$ ,  $q_0 = p^{N_0}$ , and fixed uniformiser  $t_0$ . In other words,  $\mathcal{K} = k((t_0))$ .

As earlier,  $\mathcal{G} = \text{Gal}(\mathcal{K}_{\text{sep}}/\mathcal{K})$ ,  $\mathcal{K}_{<p} = \mathcal{K}_{<p}(M)$  is the subfield of  $\mathcal{K}_{\text{sep}}$  fixed by  $\mathcal{G}^{p^M} C_p(\mathcal{G})$  and  $\mathcal{G}_{<p} = \mathcal{G}_{<p}(M) = \text{Gal}(\mathcal{K}_{<p}/\mathcal{K})$ . The ramification filtration of  $\mathcal{G}_{<p}$  was studied in details in [1, 2, 3]. We overview these results in the next subsections.

**1.1. Compatible system of lifts modulo  $p^M$ .** The uniformizer  $t_0$  of  $\mathcal{K}$  gives a  $p$ -basis for any separable extension  $\mathcal{E}$  of  $\mathcal{K}$ , i.e.  $\{1, t_0, \dots, t_0^{p-1}\}$  is a basis of the  $\mathcal{E}^p$ -module  $\mathcal{E}$ . We can use  $t_0$  to construct a functorial on  $\mathcal{E}$  (and on  $M$ ) system of lifts  $O(\mathcal{E}) (= O_M(\mathcal{E}))$  of  $\mathcal{E}$  modulo  $p^M$ . Recall that these lifts appear in the form  $W_M(\sigma^{M-1}\mathcal{E})[t]$ , where  $W_M$  is the functor of Witt vectors of length  $M$ ,  $\sigma$  is the Frobenius morphism of taking  $p$ -th power and  $t = (t_0, 0, \dots, 0) \in W_M(\mathcal{K})$ .

Note that  $t \in O(\mathcal{K}) \subset W_M(\mathcal{K})$ ,  $t \bmod p = t_0$  and  $\sigma t = t^p$ . The lift  $O(\mathcal{K})$  is naturally identified with the algebra of formal Laurent series  $W_M(k)((t))$  in the variable  $t$  with coefficients in  $W_M(k)$ . A lift  $\sigma$  of the absolute Frobenius endomorphism of  $\mathcal{K}$  to  $O(\mathcal{K})$  is uniquely determined by the condition  $\sigma t = t^p$ . For a separable extension  $\mathcal{E}$  of  $\mathcal{K}$  we then have

an extension of the Frobenius  $\sigma$  from  $\mathcal{E}$  to  $O(\mathcal{E}) (= W_M(\sigma^{M-1}\mathcal{E})[t])$ . As a result, we obtain a compatible system of lifts of the Frobenius endomorphism of  $\mathcal{K}_{sep}$  to  $O(\mathcal{K}_{sep}) = \varinjlim_{\mathcal{E}} O(\mathcal{E})$ . For simplicity, we shall

denote this lift also by  $\sigma$ . Note that  $\sigma$  is induced by the standard Frobenius endomorphism  $W_M(\sigma)$  of  $W_M(\mathcal{K}_{sep}) \supset O(\mathcal{K}_{sep})$ .

Suppose  $\eta_0 \in \text{Aut } \mathcal{K}$  and let  $W_M(\eta_0)$  be the induced automorphism of  $W_M(\mathcal{K})$ . If  $W_M(\eta_0)(t) \in O(\mathcal{K})$  then  $\eta := W_M(\eta_0)|_{O(\mathcal{K})}$  is a lift of  $\eta_0$  to  $O(\mathcal{K})$ , i.e.  $\eta \in \text{Aut } O(\mathcal{K})$  and  $\eta \bmod p = \eta_0$ . With the above notation and assumption (in particular,  $\eta(t) \in O(\mathcal{K})$ ) we have even more.

**Proposition 1.1.** *Suppose  $\mathcal{E}$  is separable over  $\mathcal{K}$ ,  $\eta_{\mathcal{E}0} \in \text{Aut } \mathcal{E}$  and  $\eta_{\mathcal{E}0}|_{\mathcal{K}} = \eta_0$ . Then  $\eta_{\mathcal{E}} := W_M(\eta_{\mathcal{E}0})|_{O(\mathcal{E})}$  is a lift of  $\eta_{\mathcal{E}0}$  to  $O(\mathcal{E})$  such that  $\eta_{\mathcal{E}}|_{O(\mathcal{K})} = \eta$ .*

*Proof.* Indeed, using that  $O(\mathcal{E}) = W_M(\sigma^{M-1}\mathcal{E})[t]$ , we obtain

$\eta_{\mathcal{E}}(W_M(\sigma^{M-1}\mathcal{E})) = W_M(\eta_{\mathcal{E}0})(W_M(\sigma^{M-1}\mathcal{E})) \subset W_M(\sigma^{M-1}\mathcal{E}) \subset O(\mathcal{E})$ ,  
and  $\eta_{\mathcal{E}}(t) = W_M(\eta_{\mathcal{E}0})(t) = W_M(\eta_0)(t) \in O(\mathcal{K}) \subset O(\mathcal{E})$ . So,  $\eta_{\mathcal{E}}(O(\mathcal{E})) \subset O(\mathcal{E})$ . Obviously,  $\eta_{\mathcal{E}} \bmod p = \eta_{\mathcal{E}0}$ .  $\square$

**Remark.** The above lifts  $\eta_{\mathcal{E}}$  commute with  $\sigma$  if and only if  $\eta$  commutes with  $\sigma$ , i.e.  $\sigma(\eta(t)) = \eta(t^p)$ . In particular, if  $\eta(t) = t\alpha^{p^{M-1}}$  with  $\alpha \in O(\mathcal{K})$  then  $\sigma(\eta(t)) = t^p\alpha^{p^M} = \eta(t^p)$  (use that  $\sigma(\alpha) \equiv \alpha^p \bmod pO(\mathcal{K})$ ).

A very special case of the above proposition appears as the following property:

— if  $\mathcal{E}/\mathcal{K}$  is Galois then the elements  $g$  of the group  $\text{Gal}(\mathcal{E}/\mathcal{K})$  can be naturally lifted to (commuting with  $\sigma$ ) automorphisms of  $O(\mathcal{E})$  via setting  $g(t) = t$ . Therefore,  $O(\mathcal{K}_{sep})$  has a natural structure of a  $\mathcal{G}$ -module, the action of  $\mathcal{G}$  commutes with  $\sigma$ ,  $O(\mathcal{K}_{sep})^{\mathcal{G}} = O(\mathcal{K})$  and  $O(\mathcal{K}_{sep})|_{\sigma=\text{id}} = W_M(\mathbb{F}_p)$ .

Everywhere below we shall use the following simplified notation.

**Notation.** If  $\mathfrak{M}$  is a  $\mathbb{Z}/p^M$ -module and  $\mathcal{E}$  is a separable extension of  $\mathcal{K}$  we set  $\mathfrak{M}_{\mathcal{E}} := \mathfrak{M}_{O(\mathcal{E})} (= \mathfrak{M} \otimes_{\mathbb{Z}/p^M} O(\mathcal{E}))$ . Similarly, we agree that  $\mathfrak{M}_k := \mathfrak{M} \otimes_{\mathbb{Z}/p^M} W_M(k)$ .

**1.2. Categories of  $p$ -groups and Lie  $\mathbb{Z}/p^M$ -algebras, [11, 12].** If  $L$  is a Lie  $\mathbb{Z}/p^M$ -algebra of nilpotent class  $< p$ , denote by  $G(L)$  the  $p$ -group obtained from  $L$  via the Campbell-Hausdorff composition law  $\circ$  defined for  $l_1, l_2 \in L$  via  $\widetilde{\exp}(l_1 \circ l_2) = \widetilde{\exp}l_1 \cdot \widetilde{\exp}l_2$ . Here

$$\widetilde{\exp}(x) = 1 + x + \cdots + x^{p-1}/(p-1)!$$

is the truncated exponential from  $L$  to the quotient of the enveloping algebra  $\mathcal{A}$  of  $L$  modulo the  $p$ -th power of its augmentation ideal  $J$ . (This construction of the Campbell-Hausdorff operation was introduced in [1], Subsection 1.2.)

The correspondence  $L \mapsto G(L)$  induces equivalence of the categories of finite Lie  $\mathbb{Z}/p^M$ -algebras and finite  $p$ -groups of exponent  $p^M$  of the same nilpotent class  $1 \leq s_0 < p$ . This equivalence can be extended to the similar categories of profinite Lie algebras and groups.

**1.3. Witt pairing and Hilbert symbol, [6, 9].** Let

$$E(\alpha, X) = \exp \left( \alpha X + \frac{\sigma(\alpha)X^p}{p} + \cdots + \frac{\sigma^n(\alpha)X^{p^n}}{p^n} \dots \right) \in W(k)[[X]],$$

where  $\alpha \in W(k)$ , be the Shafarevich version of the Artin-Hasse exponential. Set  $\mathbb{Z}^+(p) = \{a \in \mathbb{N} \mid \gcd(a, p) = 1\}$ . Then any element  $u \in \mathcal{K}^* \bmod \mathcal{K}^{*p^M}$  can be uniquely written as

$$u = t_0^{a_0} \prod_{a \in \mathbb{Z}^+(p)} E(\alpha_a, t_0^a)^{1/a \bmod \mathcal{K}^{*p^M}},$$

where  $a_0 = a_0(u) \in \mathbb{Z} \bmod p^M$  and all  $\alpha_a = \alpha_a(u) \in W(k) \bmod p^M$ .

Let  $\mathfrak{M}$  be a profinite free  $W_M(k)$ -module with the set of generators  $\{D_0\} \cup \{D_{an} \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}$ . Use the correspondences

$$(1.1) \quad t_0 \mapsto D_0, \quad E(\alpha, t_0^a)^{1/a} \mapsto \sum_{n \bmod N_0} \sigma^n(\alpha) D_{an},$$

to identify  $\mathcal{K}^*/\mathcal{K}^{*p^M}$  with a closed  $\mathbb{Z}/p^M$ -submodule in  $\mathfrak{M}$ . Under this identification we have  $\mathcal{K}^*/\mathcal{K}^{*p^M} \otimes_{\mathbb{Z}/p^M} W_M(k) = \mathfrak{M}$ .

Define the continuous action of the group  $\langle \sigma \rangle = \text{Gal}(k/\mathbb{F}_p)$  on  $\mathfrak{M}$  as an extension of the natural action on  $W_M(k)$  by setting  $\sigma D_0 = D_0$  and  $\sigma D_{an} = D_{a, n+1}$ . Then  $\mathcal{K}^*/\mathcal{K}^{*p^M} = \mathfrak{M}^{\text{Gal}(k/\mathbb{F}_p)}$ .

The Witt pairing

$$O(\mathcal{K})/(\sigma - \text{id})O(\mathcal{K}) \times \mathcal{K}^*/\mathcal{K}^{*p^M} \longrightarrow \mathbb{Z}/p^M,$$

is given explicitly by the symbol  $[f, g] = \text{Tr}(\text{Res}(f d_{\log} \text{Col } g))$ . Here  $\text{Tr} : W_M(k) \longrightarrow \mathbb{Z}/p^M$  is induced by the trace of the field extension  $k/\mathbb{F}_p$ ,  $f \in O(\mathcal{K})$  and  $\text{Col } g$  is the image of  $g \in \mathcal{K}^*/\mathcal{K}^{*p^M}$  under the group homomorphism  $\text{Col} : \mathcal{K}^*/\mathcal{K}^{*p^M} \longrightarrow O_M^*(\mathcal{K})$  uniquely defined on the above free generators of  $\mathcal{K}^*/\mathcal{K}^{*p^M}$  via the conditions  $t_0 \mapsto t$  and  $E(\alpha, t_0^a) \mapsto E(\alpha, t^a)$ . The Witt pairing is non-degenerate and determines the identification

$$\mathcal{K}^*/\mathcal{K}^{*p^M} = \text{Hom}_{\text{cont}}(O(\mathcal{K})/(\sigma - \text{id})O(\mathcal{K}), \mathbb{Z}/p^M).$$

It also coincides with the Hilbert symbol (in the case of local fields of characteristic  $p$ ) and allows us to specify explicitly the reciprocity map  $\kappa : \mathcal{K}^*/\mathcal{K}^{*p^M} \longrightarrow \mathcal{G}_{<p}^{ab}$  of class field theory. Namely, in the above notation we have  $\kappa(g)f = f + [f, g]$ .

**1.4. Lie algebra  $\mathcal{L}$  and identification  $\eta_M$ .** Let  $\tilde{\mathcal{L}}$  be a free profinite Lie  $\mathbb{Z}/p^M$ -algebra with the module of (free) generators  $\mathcal{K}^*/\mathcal{K}^{*p^M}$ . Then the  $W_M(k)$ -module  $\tilde{\mathcal{L}}_k$  has the set of free generators

$$(1.2) \quad \{D_0\} \cup \{D_{an} \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}.$$

If  $C_p(\tilde{\mathcal{L}})$  is the closure of the ideal of commutators of order  $\geq p$ , then  $\mathcal{L} = \tilde{\mathcal{L}}/C_p(\tilde{\mathcal{L}})$  is the maximal quotient of  $\tilde{\mathcal{L}}$  of nilpotent class  $< p$ .

**Remark.**  $\mathcal{L}_k$  is a free object in the category of profinite Lie  $W_M(k)$ -algebras of nilpotent class  $< p$  with the set of free generators (1.2).

We shall use the same notation  $D_0$  and  $D_{an}$  for the images of the elements of (1.2) in  $\mathcal{L}$ . Choose  $\alpha_0 \in W_M(k)$  such that  $\text{Tr } \alpha_0 = 1$ .

Consider  $e = \alpha_0 D_0 + \sum_{a \in \mathbb{Z}^+(p)} t^{-a} D_{a0} \in G(\mathcal{L}_K)$ . If we set  $D_{0n} := (\sigma^n \alpha_0) D_0$  then  $e$  can be written as  $\sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0}$ , where  $\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}$ .

Fix  $f \in G(\mathcal{L}_{K_{sep}})$  such that  $\sigma f = e \circ f$ . Then for  $\tau \in \mathcal{G}$ , the correspondence

$$\tau \mapsto (-f) \circ \tau f \in G(\mathcal{L}_{K_{sep}})|_{\sigma=\text{id}} = G(\mathcal{L}),$$

induces the identification of profinite groups  $\eta_M : \mathcal{G}_{<p} \simeq G(\mathcal{L})$ .

Note that  $f \in \mathcal{L}_{K_{<p}}$  and  $\mathcal{G}_{<p}$  strictly acts on the  $\mathcal{G}$ -orbit of  $f$ .

The above result is a covariant version of the nilpotent Artin-Schreier theory developed in [2], cf. also Subsection 1.1 in [7] for the relation between the covariant and contravariant versions of this theory and for appropriate non-formal comments.

We shall use below a fixed choice of  $f$  and use the notation for  $e$  and  $f$  without further references.

**1.5. Relation to class field theory.** The above identification  $\eta_M$  taken modulo  $C_2(\mathcal{G}_{<p})$  gives an isomorphism of profinite  $p$ -groups

$$\eta_M^{ab} : \mathcal{G}_{<p}^{ab} \longrightarrow \mathcal{L}^{ab} = \mathcal{L}/C_2(\mathcal{L}) = \mathfrak{M}^{\text{Gal}(k/\mathbb{F}_p)} = \mathcal{K}^*/\mathcal{K}^{*p^M}.$$

**Proposition 1.2.**  $\eta_M^{ab}$  is induced by the inverse to the reciprocity map of local class field theory  $\kappa$ .

*Proof.* Indeed, let  $\{\beta_i\}_{1 \leq i \leq N_0}$  be a  $\mathbb{Z}/p^M$ -basis of  $W_M(k)$  and let  $\{\gamma_i\}_{1 \leq i \leq N_0}$  be its dual basis with respect to the bilinear form induced by the trace of the field extension  $W(k)[1/p]/\mathbb{Q}_p$ .

If  $a \in \mathbb{Z}^+(p)$  and  $E(\beta_i, t_0^a)^{1/a} = D_{ia}$ , then  $D_{ia} = \sum_n \sigma^n(\beta_i) D_{an}$ , and, therefore,  $D_{a0} = \sum_i \gamma_i D_{ia}$ . This implies that

$$e = \sum_{i,a} t^{-a} \gamma_i D_{ia} + \alpha_0 D_0 \bmod C_2(\mathcal{L}_K),$$

$$f = \sum_{i,a} f_{ia} D_{ia} + f_0 D_0 \bmod C_2(\mathcal{L}_{K_{sep}}),$$

where all  $f_{ia}, f_0 \in O(\mathcal{K}_{< p})$ ,  $\sigma f_{ia} - f_{ia} = \gamma_i t^{-a}$  and  $\sigma f_0 - f_0 = \alpha_0$ . From the definition of  $\eta_M$  it follows formally that for  $\tau_{ia} = (\eta_M^{ab})^{-1} D_{ia}$  and  $\tau_0 = (\eta_M^{ab})^{-1} D_0$ ,  $\tau_{ia} f_{i_1 a_1} = f_{i_1 a_1} + \delta(ii_1) \delta(aa_1)$ ,  $\tau_0 f_{i_1 a_1} = f_{i_1 a_1}$ ,  $\tau_{ia} f_0 = f_0$  and  $\tau_0 f_0 = f_0 + 1$ . (Here  $\delta$  is the Kronecker symbol.)

Now the explicit formula for the Hilbert symbol from Subsection 1.3 shows that  $\kappa(E(\beta_i, t_0^{1/a}))$  and  $\kappa(t_0)$  act by the same formulae as  $\tau_{ia}$  and, resp.,  $\tau_0$ .  $\square$

**1.6. Construction of lifts of analytic automorphisms.** Let  $\eta_0 \in \text{Aut } \mathcal{K}$ . Then there is a lift  $\eta_{< p, 0} \in \text{Aut } \mathcal{K}_{< p}$  of  $\eta_0$ . (Use that the subgroup  $\mathcal{G}^{p^M} C_p(\mathcal{G})$  of  $\mathcal{G}$  is characteristic.) For any another such lift  $\eta'_{< p, 0}$ , we have  $\eta'_{< p, 0} \eta_{< p, 0}^{-1} \in \mathcal{G}_{< p}$ .

The covariant version of the Witt-Artin-Schreier theory [2], Section 1 (cf. also [7], Subsection 1.1 and [8], Section 1), gives explicit description of the automorphisms  $\eta_{< p, 0}$  in terms of the identification  $\eta_M$ . Consider a special case of this construction when  $\eta_0$  admits a lift  $\eta \in \text{Aut } O(\mathcal{K})$  which commutes with  $\sigma$ , and therefore we have the appropriate lifts  $\eta_{< p} \in \text{Aut } O(\mathcal{K}_{< p})$ , cf. Subsection 1.1. Then in terms of our fixed elements  $e$  and  $f$ , we have  $\eta_{< p}(f) = c \circ (A \otimes \text{id}_{O(\mathcal{K}_{< p})})f$ , where  $c \in \mathcal{L}_{\mathcal{K}}$  and  $A \in \text{Aut } \mathcal{L}$  can be found from the relation

$$(\text{id}_{\mathcal{L}} \otimes \eta)e = \sigma c \circ (A \otimes \text{id}_{O(\mathcal{K})})e \circ (-c),$$

cf. Subsection 1.5 in [2], or Proposition 1.1 in [8], and Subsection 3.2 below.

In other words, if  $(A \otimes \text{id}_{W_M(k)})(D_{a0}) = \tilde{D}_{a0}$  then

$$\sum_{a \in \mathbb{Z}^0(p)} \eta(t)^{-a} D_{a0} = \sigma c \circ \left( \sum_{a \in \mathbb{Z}^0(p)} t^{-a} \tilde{D}_{a0} \right) \circ (-c).$$

Note that proceeding as in [2], Subsection 1.5.4, cf. also [8], Subsection 1.2, we can verify (this fact will be used systematically below) that with respect to the identification  $\eta_M$ , the automorphism  $A$  coincides with the conjugation  $\text{Ad } \eta_{< p} : \tau \mapsto \eta_{< p}^{-1} \tau \eta_{< p}$  (here  $\tau \in \mathcal{G}_{< p}$ ).

**1.7. Ramification filtration in  $\mathcal{L}$ .** For  $v \geq 0$ , denote by  $\mathcal{G}_{< p}^{(v)}$  the ramification subgroup of  $\mathcal{G}_{< p}$  with the upper index  $v$ . Let  $\mathcal{L}^{(v)}$  be the ideal of  $\mathcal{L}$  such that  $\eta_M(\mathcal{G}_{< p}^{(v)}) = G(\mathcal{L}^{(v)})$ . The ideals  $\mathcal{L}^{(v)}$  have the following explicit description.

First, for any  $a \in \mathbb{Z}^0(p)$  and  $n \in \mathbb{Z}$ , set  $D_{an} := D_{a, n \bmod N_0}$ . In other words, we allow the second index in all  $D_{an}$  to take integral values and assume that  $D_{an_1} = D_{an_2}$  iff  $n_1 \equiv n_2 \bmod N_0$ . For  $s \geq 1$ , agree to use the notation  $(\bar{a}, \bar{n})_s$ , where  $\bar{a} = (a_1, \dots, a_s)$  has coordinates in  $\mathbb{Z}^0(p)$  and  $\bar{n} = (n_1, \dots, n_s) \in \mathbb{Z}^s$ . Then we can attach to  $(\bar{a}, \bar{n})_s$  the commutator

$[\dots [D_{a_1 n_1}, D_{a_2 n_2}], \dots, D_{a_s n_s}]$  and set  $\gamma(\bar{a}, \bar{n})_s = a_1 p^{n_1} + \dots + a_s p^{n_s}$ . For any  $\gamma \geq 0$ , let  $\mathcal{F}_{\gamma, -N}^0$  be the element from  $\mathcal{L}_k$  given by

$$(1.3) \quad \mathcal{F}_{\gamma, -N}^0 = \sum_{\gamma(\bar{a}, \bar{n})_s = \gamma} p^{n_1} a_1 \eta(\bar{n}) [\dots [D_{a_1 n_1}, D_{a_2 n_2}], \dots, D_{a_s n_s}]$$

where  $\eta(\bar{n})$  equals  $(s_1!(s_2 - s_1)! \dots (s_l - s_{l-1})!)^{-1}$  if  $0 \leq n_1 = \dots = n_{s_1} > n_{s_1+1} = \dots = n_{s_2} > \dots > n_{s_l} = \dots = n_s \geq -N$ , and equals to zero otherwise. Then the main result of [3] (translated into the covariant setting, cf. [4], Subsections 1.1.2 and 1.2.4) states that

• *there is  $\tilde{N}(v) \in \mathbb{N}$  such that if we fix any  $N \geq \tilde{N}(v)$ , then  $\mathcal{L}^{(v)}$  is the minimal ideal of  $\mathcal{L}$  such that for all  $\gamma \geq v$ ,  $\mathcal{F}_{\gamma, -N}^0 \in \mathcal{L}_k^{(v)}$ .*

## 2. FILTRATION $\{\mathcal{L}(s)\}_{s \geq 1}$

In this section we define a decreasing central filtration  $\{\mathcal{L}(s)\}_{s \geq 1}$  in the  $\mathbb{Z}/p^M$ -Lie algebra  $\mathcal{L}$  from Subsection 1.4. Its definition depends on a choice of a special element  $S \in \mathfrak{m}(\mathcal{K}) := tW_M(k)[[t]] \subset O(\mathcal{K})$ . This element  $S$  (together with the appropriate elements  $S_0$  and  $S'$  from its definition) will be specified in Section 4, where we apply our results to the mixed characteristic case.

### 2.1. Elements $S_0, S', S \in \mathfrak{m}(\mathcal{K})$ .

Let  $[p]$  be the isogeny of multiplication by  $p$  in the formal group  $\mathrm{Spf} \mathbb{Z}_p[[X]]$  over  $\mathbb{Z}_p$  with the logarithm  $X + X^p/p + \dots + X^{p^n}/p^n + \dots$ .

Choose  $S_0 \in \mathfrak{m}(\mathcal{K})$  and set  $S' = [p]^{M-1}(S_0)$  and  $S = [p]^M(S_0)$ . Then  $S, S' \in \mathfrak{m}(\mathcal{K})$ , they both depend only on the residue  $S_0 \bmod p$  and  $S = \sigma S'$ . In particular, if  $e^* \in \mathbb{N}$  is such that  $S \bmod p$  generates the ideal  $(t_0^{e^*})$  in  $k[[t_0]]$  then  $e^* \equiv 0 \bmod p^M$ .

**Proposition 2.1.** a)  $dS = 0$  in  $\Omega_{O(\mathcal{K})}^1$ ;

b) *there is  $S'' \in \mathfrak{m}(\mathcal{K})$ , such that  $S = S'(p + S'')$ ;*

c) *there are  $\eta_0, \eta_1 \in W_M(k)[[t]]^\times$  and  $\eta_2 \in W_M(k)[[t]]$  such that*

$$S = t^{e^*} \eta_0 + p t^{e^*/p} \eta_1 + p^2 \eta_2.$$

*Proof.* a) The congruence  $[p]X \equiv X^p \bmod p\mathbb{Z}_p[[X]]$  implies that  $d([p]X) \in p\mathbb{Z}_p[[X]]$ . Therefore,  $dS = 0$  in  $\Omega_{O(\mathcal{K})}^1$ .

b) Note that  $[p](X) \equiv pX \bmod X^2$ . Therefore, there are  $w_i \in \mathbb{Z}_p$  such that  $S = [p]S' = pS' + \sum_{i \geq 2} w_i S'^i$  and we can take  $S'' = \sum_{i \geq 1} w_{i+1} S'^i$ .

c) The  $t_0$ -adic valuation of  $S' \bmod p$  equals  $e^*/p$ . Then our property is implied by the following equivalence in  $\mathbb{Z}_p[[X]]$

$$[p](X) \equiv pX + X^p \bmod (pX^{p^2-p+1}, p^2X).$$

□



**Remark.** We shall use below property a) in the following form:

if  $s \in \mathbb{N}$  and  $S^s = \sum_{l \geq 1} \gamma_{ls} t^l$ , where all  $\gamma_{ls} \in W_M(k)$ , then  $l\gamma_{ls} = 0$ .

**2.2. Morphism  $\iota$ .** Let  $\mathcal{U} = (1 + t_0 k[[t_0]])^\times$  be the  $\mathbb{Z}_p$ -module of principal units in  $\mathcal{K}$ . Then  $\mathcal{U}/\mathcal{U}^{p^M}$  is a closed  $\mathbb{Z}/p^M$ -submodule in  $\mathcal{K}^*/\mathcal{K}^{*p^M}$ . Note that  $\mathfrak{m}(\mathcal{K}) = W_M(\mathfrak{m}_{\mathcal{K}}) \cap O(\mathcal{K})$ , where  $\mathfrak{m}_{\mathcal{K}}$  is the maximal ideal in the valuation ring of  $\mathcal{K}$ . Consider a (unique) continuous homomorphism

$$\iota : \mathcal{U} \longrightarrow \mathfrak{m}(\mathcal{K})$$

such that for any  $\alpha \in W_M(k)$  and  $a \in \mathbb{Z}^+(p)$ ,  $\iota : E(\alpha, t_0^a) \mapsto \alpha t^a$  (here  $E$  is the Shafarevich function, cf. Subsection 1.3).

Then  $\iota$  induces an identification of  $\mathcal{U}/\mathcal{U}^{p^M}$  with the closed  $W_M(k)$ -submodule

$$\text{Im } \iota = \left\{ \sum_{a \in \mathbb{Z}^+(p)} \alpha_a t^a \mid \alpha_a \in W_M(k) \right\}$$

in  $O(\mathcal{K})$ . This submodule is topologically generated over  $W_M(k)$  by all  $t^a$  with  $a \in \mathbb{Z}^+(p)$ .

**2.3. Definition of  $\{\mathcal{L}(s)\}_{s \geq 1}$ .** Set  $(\mathcal{K}^*/\mathcal{K}^{*p^M})^{(1)} = \mathcal{K}^*/\mathcal{K}^{*p^M}$ . For  $s \geq 1$ , let  $(\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s+1)} = (\text{Im } \iota)S^s$  with respect to the identification  $\mathcal{U}/\mathcal{U}^{p^M} = \text{Im } \iota$  from Subsection 2.2. Note, that  $S = \sigma S'$  implies that for any  $s \in \mathbb{N}$ ,  $(\text{Im } \iota)S^s \subset \text{Im } \iota$ .

**Definition.**  $\{\mathcal{L}(s)\}_{s \geq 1}$  is the minimal central filtration of ideals of the Lie algebra  $\mathcal{L}$  such that for all  $s \geq 1$ ,  $\mathcal{L}(s) \supset (\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)}$ .

The ideals  $\mathcal{L}(s)$  can be defined by induction on  $s$  as follows. Let  $\mathcal{L}(1) = \mathcal{L}$ ; then for  $s \geq 1$ , the ideal  $\mathcal{L}(s+1)$  is generated by the elements of  $(\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s+1)}$  and  $[\mathcal{L}(s), \mathcal{L}]$ . Note also that for any  $s$ ,  $(\mathcal{K}^*/\mathcal{K}^{*p^M}) \cap \mathcal{L}(s) = (\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)}$ . (Use that  $\mathbb{Z}/p^M$ -module  $\mathcal{L}(s)$  is isomorphic to  $(\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)} \oplus (\mathcal{L}(s) \cap C_2(\mathcal{L}))$ ).

In addition, for any  $s \geq 1$ , the quotients  $(\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)}/(\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s+1)}$  are free  $\mathbb{Z}/p^M$ -modules. This easily implies that all  $\mathcal{L}(s)/\mathcal{L}(s+1)$  are also free  $\mathbb{Z}/p^M$ -modules.

**2.4. Characterization of  $\{\mathcal{L}(s)\}_{s \geq 1}$  in terms of  $e \in \mathcal{L}_{\mathcal{K}}$ .** Recall that  $e = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0}$ , cf. Subsection 1.4.

**Proposition 2.2.** *The filtration  $\{\mathcal{L}(s)\}_{s \geq 1}$  is the minimal central filtration in  $\mathcal{L}$  such that  $\mathcal{L}(1) = \mathcal{L}$  and for all  $s \geq 1$ ,*

$$S^s e \in \mathcal{L}_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(s+1)_{\mathcal{K}}.$$

*Proof.* We need the following two lemmas.

**Lemma 2.3.** *For all  $s \geq 1$  and  $\alpha_a \in W_M(k)$  where  $a \in \mathbb{Z}^+(p)$ , we have*

$$\prod_{a \in \mathbb{Z}^+(p)} E(\alpha_a, t_0^a) \in (\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s+1)} \Leftrightarrow \prod_{a \in \mathbb{Z}^+(p)} E(\alpha_a, t_0^a)^{1/a} \in (\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s+1)}.$$

*Proof of Lemma.* We must prove that

$$\sum_{a \in \mathbb{Z}^+(p)} \alpha_a t^a \in S^s \mathfrak{m}(\mathcal{K}) \Leftrightarrow \sum_{a \in \mathbb{Z}^+(p)} \frac{1}{a} \alpha_a t^a \in S^s \mathfrak{m}(\mathcal{K}).$$

Let  $S^s = \sum_{l \geq 1} \gamma_{ls} t^l$  with  $\gamma_{ls} \in W_M(k)$ , then  $l\gamma_{ls} = 0$ , cf. Remark in Subsection 2.1.

Suppose  $\sum_{a \in \mathbb{Z}^+(p)} \alpha_a t^a \in S^s \mathfrak{m}(\mathcal{K})$ .

Then  $\sum_a \alpha_a t^a = (\sum_b \beta_b t^b)(\sum_l \gamma_{ls} t^l)$ , where  $\sum_b \beta_b t^b \in \mathfrak{m}(\mathcal{K})$  and  $\alpha_a = \sum_{a=b+l} \beta_b \gamma_{ls}$ . This implies

$$\frac{1}{a} \alpha_a = \sum_{a=b+l} \frac{1}{a} \beta_b \gamma_{ls} = \sum_{a=b+l} \frac{1}{b} \beta_b \gamma_{ls},$$

because if  $a = b + l$  and  $a \in \mathbb{Z}^+(p)$  then  $b \in \mathbb{Z}^+(p)$  and

$$\frac{1}{a} \gamma_{ls} - \frac{1}{b} \gamma_{ls} = \frac{-l\gamma_{ls}}{ab} = 0.$$

So,

$$\sum_{a \in \mathbb{Z}^+(p)} \frac{1}{a} \alpha_a t^a = \left( \sum_{b \in \mathbb{Z}^+(p)} \frac{1}{b} \beta_b t^b \right) \left( \sum_l \gamma_{ls} t^l \right)$$

and  $\sum_a \frac{1}{a} \alpha_a t^a \in S^s \mathfrak{m}(\mathcal{K})$ .

Proceeding in the opposite direction we obtain the inverse statement. The lemma is proved.  $\square$

**Lemma 2.4.** *If  $s \geq 1$  and all  $\alpha_a \in W_M(k)$  then*

$$\prod_{a \in \mathbb{Z}^+(p)} E(\alpha_a, t_0^a)^{1/a} \in (\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)} \Leftrightarrow \sum_{a \in \mathbb{Z}^+(p)} \alpha_a D_{a0} \in (\mathcal{K}^*/\mathcal{K}^{*p^M})_k^{(s)}$$

*Proof of Lemma.* Suppose

$$\prod_{a \in \mathbb{Z}^+(p)} E(\alpha_a, t_0^a)^{1/a} \in (\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)}.$$

Choose a  $W_M(\mathbb{F}_p)$ -basis  $\{\beta_i\}$  of  $W_M(k)$ , and let  $\{\gamma_i\}$  be its dual with respect to the trace form. Then for any  $i$ ,

$$\prod_{a \in \mathbb{Z}^+(p)} E(\beta_i \alpha_a, t_0^a)^{1/a} \in (\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)}.$$

In other words (use (1.1) from Subsection 1.3),

$$c_i = \sum_{\substack{a \in \mathbb{Z}^+(p) \\ n \in \mathbb{Z}/N_0\mathbb{Z}}} \sigma^n(\beta_i) \sigma^n(\alpha_a) D_{an} \in \left( \mathcal{K}^*/\mathcal{K}^{*p^M} \right)^{(s)} \subset \mathcal{L}(s),$$

and

$$\sum_i \gamma_i c_i = \sum_{a \in \mathbb{Z}^+(p)} \alpha_a D_{a0} \in \mathcal{L}(s)_k.$$

Suppose now that  $\sum_{a \in \mathbb{Z}^+(p)} \alpha_a D_{a0} \in \mathcal{L}(s)_k$ . Then

$$\sum_{a \in \mathbb{Z}^+(p)} \alpha_a D_{a0} \in (\mathcal{K}^* / \mathcal{K}^{*p^M})_k^{(s)},$$

and, therefore,

$$\sum_{\substack{a \in \mathbb{Z}^+(p) \\ n \in \mathbb{Z}/N_0\mathbb{Z}}} \sigma^n(\alpha_a) D_{an} \in (\mathcal{K}^* / \mathcal{K}^{*p^M})^{(s)}.$$

This means, that

$$\prod_{a \in \mathbb{Z}^+(p)} E(\alpha_a, t_0^a)^{1/a} \in (\mathcal{K}^* / \mathcal{K}^{*p^M})^{(s)}.$$

The lemma is proved.  $\square$

Now we can finish the proof of our proposition. If, as earlier,  $S^s = \sum_{l \geq 1} \gamma_{ls} t^l$  with  $\gamma_{ls} \in W_M(k)$ , then  $(\text{Im } \iota) S^s$  is the  $W_M(k)$ -submodule in  $\mathfrak{m}(\mathcal{K})$  generated by the elements  $t^{a_1} S^s = \sum_{l \geq 1} \gamma_{ls} t^{l+a_1}$ ,  $a_1 \in \mathbb{Z}^+(p)$ . The above lemmas imply then that  $\{\mathcal{L}(s)\}_{s \geq 1}$  is the minimal central filtration in  $\mathcal{L}$  such that  $\mathcal{L}(1) = \mathcal{L}$  and for all  $a_1 \in \mathbb{Z}^+(p)$ ,  $s \geq 1$ ,

$$\sum_{l \geq 1} \gamma_{ls} D_{a_1+l,0} \in \mathcal{L}(s+1)_k.$$

On the other hand,

$$S^s e = \sum_{\substack{a \in \mathbb{Z}^0(p) \\ l \geq 1}} \gamma_{ls} t^{-(a-l)} D_{a0} \equiv \sum_{a_1 \in \mathbb{Z}^+(p)} \left( \sum_{l \geq 1} \gamma_{ls} D_{a_1+l,0} \right) t^{-a_1}$$

modulo  $\mathcal{L}_{\mathfrak{m}(\mathcal{K})}$ . Therefore,

$$S^s e \in \mathcal{L}_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(s+1)_{\mathcal{K}} \Leftrightarrow$$

$$\sum_l \gamma_{ls} D_{a_1+l,0} \in \mathcal{L}(s+1)_k \text{ for all } a_1 \in \mathbb{Z}^+(p).$$

The proposition is proved.  $\square$

**Definition.**  $\mathcal{N} = \sum_{s \geq 1} S^{-s} \mathcal{L}(s)_{\mathfrak{m}(\mathcal{K})}$ .

Note that  $\mathcal{N}$  is a Lie  $W_M(\mathbb{F}_p)$ -subalgebra in  $\mathcal{L}_{\mathcal{K}}$ . With this notation Proposition 2.2 implies the following characterization of the filtration  $\{\mathcal{L}(s)\}_{s \geq 1}$ .

**Corollary 2.5.**  $\{\mathcal{L}(s)\}_{s \geq 1}$  is the minimal central filtration in  $\mathcal{L}$  such that  $\mathcal{L}(1) = \mathcal{L}$  and  $e \in \mathcal{N}$ .

*Proof.* It will be sufficient to verify that

$$e \in \mathcal{N} \Leftrightarrow \forall s \geq 1, S^s e \in \mathcal{L}_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(s+1)_{\mathcal{K}}.$$

The “if” part is obvious. The “only if” part can be proved by induction on  $s$  via the following property:

— if  $l'(s) \in \mathcal{L}(s)_{\mathcal{K}}$  and  $Sl'(s) \in \mathcal{L}_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(s+1)_{\mathcal{K}}$  then  $l'(s) \in S^{-1}\mathcal{L}(s)_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(s+1)_{\mathcal{K}}$  (use that  $\mathcal{L}(s)/\mathcal{L}(s+1)$  is free  $\mathbb{Z}/p^M$ -module).  $\square$

**2.5. Element  $e^\dagger \in G(\mathcal{L}_{\mathcal{K}})$ .** Recall that  $S \bmod p$  generates the ideal  $(t_0^{e^*})$  in  $k[[t_0]]$ . Therefore, the projections of the elements of the set

$$\{S^{-m}t^b \mid 1 \leq b < e^*, \gcd(b, p) = 1, m \in \mathbb{N}\} \cup \{\alpha_0\}$$

form a basis of  $O(\mathcal{K})/(\sigma - \text{id})O(\mathcal{K})$  over  $W_M(k)$ .

**Proposition 2.6.** *There are  $V_{(0)} \in \mathcal{L}$ ,  $x \in S\mathcal{N}$  and  $V_{(b,m)} \in \mathcal{L}_k$ , where  $m \geq 1$ ,  $1 \leq b < e^*$ ,  $\gcd(b, p) = 1$ , such that*

$$\text{a) } e^\dagger := \sum_{m,b} S^{-m}t^b V_{(b,m)} + \alpha_0 V_{(0)} \in \mathcal{N};$$

$$\text{b) } e^\dagger = (-\sigma x) \circ e \circ x.$$

*Proof.* Note that  $S \in \sigma\mathfrak{m}(\mathcal{K})$  implies that the sets  $\{t^{-a} \mid a \in \mathbb{Z}^+(p)\}$  and  $\{S^{-m}t^b \mid m \in \mathbb{N}, \gcd(b, p) = 1, 1 \leq b < e^*\}$  generate the same  $W_M(k)$ -submodules in  $O(\mathcal{K})/\mathfrak{m}(\mathcal{K})$ . This implies the existence of  $V_{(0)}^{(0)} \in \mathcal{L}$  and  $V_{(b,m)}^{(0)} \in \mathcal{L}_k$  such that

$$(2.1) \quad e \equiv e_0^\dagger \bmod \mathcal{L}_{\mathfrak{m}(\mathcal{K})}$$

where  $e_0^\dagger := \sum_{(b,m)} S^{-m}t^b V_{(b,m)}^{(0)} + \alpha_0 V_{(0)}^{(0)}$ .

For  $i \geq 1$ , let  $\mathcal{N}^{(i)} = \sum_{s \geq i} S^{-s}\mathcal{L}(s)_{\mathfrak{m}(\mathcal{K})}$ . Then

- $\mathcal{N}^{(i)} = S^{-i}\mathcal{L}(i)_{\mathfrak{m}(\mathcal{K})} + \mathcal{N}^{(i+1)}$ ;
- $[\mathcal{N}^{(i)}, \mathcal{N}] \subset \mathcal{N}^{(i+1)}$ .

In particular, relation (2.1) implies that  $e = e_0^\dagger + \sigma x_0 - x_0$ , where  $x_0 \in \mathcal{L}_{\mathfrak{m}(\mathcal{K})}$ , and we obtain

$$(2.2) \quad (-\sigma x_0) \circ e \circ x_0 \equiv e_0^\dagger \bmod S\mathcal{N}^{(2)}$$

(use that  $x_0, \sigma x_0 \in \mathcal{L}_{\mathfrak{m}(\mathcal{K})} \subset S\mathcal{N}^{(1)}$ ). Now we need the following lemma.

**Lemma 2.7.** *Suppose  $\mathfrak{M}$  is a  $\mathbb{Z}_p$ -module and  $i_0 \in \mathbb{N}$ . Then for any  $l \in S^{-i_0}\mathfrak{M}_{\mathfrak{m}(\mathcal{K})}$ , there are  $l_{(0)} \in \mathfrak{M}$ ,  $\tilde{l} \in S^{-i_0}\mathfrak{M}_{\mathfrak{m}(\mathcal{K})}$  and  $l_{(b,m)} \in \mathfrak{M}_k$ , where  $1 \leq m \leq i_0$ ,  $\gcd(p, b) = 1$  and  $1 \leq b < e^*$ , such that*

$$l = \sum_{b,m} S^{-m}t^b l_{(b,m)} + \alpha_0 l_{(0)} + \sigma \tilde{l} - \tilde{l}.$$

*Proof of Lemma 2.7.* It will be sufficient to consider the case  $\mathfrak{M} = \mathbb{Z}_p$ . In other words, we must prove the following statement:

• For any  $s \in S^{-i_0}\mathfrak{m}(\mathcal{K})$ , there are  $\beta_{(0)} \in W_M(\mathbb{F}_p)$ ,  $\tilde{s} \in S^{-i_0}\mathfrak{m}(\mathcal{K})$  and  $\beta_{(b,m)} \in W_M(k)$ , where  $1 \leq m \leq i_0$ ,  $\gcd(b, p) = 1$  and  $1 \leq b < e^*$ , such that

$$s = \sum_{b,m} \beta_{(b,m)} S^{-m} t^b + \alpha_0 \beta_{(0)} + \sigma \tilde{s} - \tilde{s}.$$

We can assume that  $s = t^{a_0}/S^{i_0}$ , where  $1 \leq a_0 < e^*$ ,  $i_0 \in \mathbb{N}$  and our lemma is proved for all elements  $s$  from  $pS^{-i_0}\mathfrak{m}(\mathcal{K}) + t^{a_0}S^{-i_0}\mathfrak{m}(\mathcal{K})$ .

If  $\gcd(a_0, p) = 1$  there is nothing to prove. Otherwise,  $a_0 = pa_1$  and  $s = s' + \sigma(s') - s'$  with  $s' = t^{a_1}/S^{i_0} = t^{a_1}(p + S'')/S^{i_0}$ . It remains to note that  $s' \in pS^{-i_0}\mathfrak{m}(\mathcal{K}) + t^{a_0}S^{-i_0}\mathfrak{m}(\mathcal{K})$ , because  $S'' \bmod p \in (t_0^{e^0})$ , where  $e^0 := e^*(1 - 1/p)$ , and  $a_1 + e^0 = a_0/p + e^0 > a_0$  (use that  $a_0 < e^*$ ).  $\square$

Continue the proof of Proposition 2.6. Clearly, it is implied by the following lemma.

**Lemma 2.8.** For all  $i \geq 0$ , there are  $x_i \in S\mathcal{N}$ ,  $V_{(b,m)}^{(i)} \in \mathcal{L}_k$  and  $V_{(0)}^{(i)} \in \mathcal{L}$  such that:

- $a_1) \ x_{i+1} \equiv x_i \bmod S\mathcal{N}^{(i+1)};$
- $a_2) \ V_{(b,m)}^{(i+1)} \equiv V_{(b,m)}^{(i)} \bmod \mathcal{L}(i+2)_k;$
- $a_3) \ V_{(0)}^{(i+1)} \equiv V_{(0)}^{(i)} \bmod \mathcal{L}(i+2)$
- b) if  $e_i^\dagger = \sum_{b,m} S^{-m} t^b V_{(b,m)}^{(i)} + \alpha_0 V_{(0)}^{(i)}$  then

$$(-\sigma x_i) \circ e \circ x_i \equiv e_i^\dagger \bmod S\mathcal{N}^{(i+2)}.$$

*Proof.* Use the elements  $V_{(b,m)}^{(0)}, V_{(0)}^{(0)}, e_0^\dagger$  and  $x_0$  from the beginning of the proof of Proposition 2.6. Then part b) holds for  $i = 0$  by (2.2).

Let  $i_0 \geq 1$  and assume that our Lemma is proved for all  $i < i_0$ . Let  $l \in S^{-i_0}\mathcal{L}(i_0 + 1)_{\mathfrak{m}(\mathcal{K})}$  be such that

$$e_{i_0-1}^\dagger - (-\sigma x_{i_0-1}) \circ e \circ x_{i_0-1} \equiv l \bmod S\mathcal{N}^{(i_0+2)}.$$

Apply Lemma 2.7 to  $\mathfrak{M} = \mathcal{L}(i_0 + 1)$  and  $l \in S^{-i_0}\mathcal{L}(i_0 + 1)_{\mathfrak{m}(\mathcal{K})}$ . This gives us the appropriate elements  $l_{(b,m)} \in \mathcal{L}(i_0 + 1)_k$ ,  $l_{(0)} \in \mathcal{L}(i_0 + 1)$  and  $\tilde{l} \in S^{-i_0}\mathcal{L}(i_0 + 1)_{\mathfrak{m}(\mathcal{K})}$ . Note that the elements  $l_{(b,m)}$  are defined only for  $1 \leq m \leq i_0$ . Extend their definition by setting  $l_{(b,m)} = 0$  if  $m > i_0$ . Then the case  $i = i_0$  of Lemma 2.8 holds with  $V_{(b,m)}^{(i_0)} = V_{(b,m)}^{(i_0-1)} + l_{(b,m)}$ ,  $V_{(0)}^{(i_0)} = V_{(0)}^{(i_0-1)} + l_{(0)}$  and  $x_{i_0} = x_{i_0-1} + \tilde{l}$ . (We use here that  $S\mathcal{N}^{(i_0+1)} = S^{-i_0}\mathcal{L}(i_0 + 1)_{\mathfrak{m}(\mathcal{K})} + S\mathcal{N}^{(i_0+2)}$ .)

Lemma 2.8 and Proposition 2.6 are completely proved.  $\square$

$\square$

Proposition 2.6b) implies that the elements  $\sigma^n V_{(b,m)}$ ,  $n \in \mathbb{Z}/N_0$ , together with  $V_{(0)}$  form a system of free topological generators of  $\mathcal{L}_k$ . Suppose  $\{\beta_i\}_{1 \leq i \leq N_0}$  and  $\{\gamma_i\}_{1 \leq i \leq N_0}$  are the  $\mathbb{Z}/p^M$ -bases of  $W_M(k)$  from the proof of Proposition 1.2. Proceeding similarly to that proof introduce the elements

$$V_{(b,m),i} := \sum_{n \in \mathbb{Z}/N_0} \sigma^n(\beta_i) \sigma^n(V_{(b,m)}).$$

Then all  $V_{(b,m)}$  can be recovered via the relation  $V_{(b,m)} = \sum_i \gamma_i V_{(b,m),i}$ . This implies that the elements  $V_{(b,m),i}$  together with  $V_{(0)}$  form a system of free topological generators of  $\mathcal{L}$ . (Recall that  $\mathcal{L}$  is a free object in the category of Lie  $\mathbb{Z}/p^M$ -algebras of nilpotent class  $< p$ .) Therefore, we can introduce the weight function  $\text{wt}$  on  $\mathcal{L}$  by setting for all  $b, m, i$ ,  $\text{wt}(V_{(b,m),i}) = m$  and  $\text{wt}(V_{(0)}) = 1$ . Note that by Proposition 2.6b) we have that  $e^\dagger \in \mathcal{N}$  if and only if  $e \in \mathcal{N}$ . Now Proposition 2.2 implies the following corollary.

**Corollary 2.9.** *For any  $s \geq 1$ ,  $\mathcal{L}(s) = \{l \in \mathcal{L} \mid \text{wt}(l) \geq s\}$ .*

### 3. THE GROUPS $\tilde{\mathcal{G}}_h$ AND $\mathcal{G}_h$

**3.1. Automorphism  $h$ .** Let  $S \in O(\mathcal{K})$  be the element introduced in Subsection 2.1. Let  $h_0 \in \text{Aut}(\mathcal{K})$  be such that  $h_0|_k = \text{id}$  and  $h_0(t_0) = t_0 E(1, S \bmod p)$ . Then  $h_0$  admits a lift to  $h \in \text{Aut } O(\mathcal{K})$  such that  $h|_{W_M(k)} = \text{id}$  and  $h(t) = t E(1, S)$ . Recall that  $O(\mathcal{K}) = W_M(k)((t))$ . If  $n \in \mathbb{N}$  then denote by  $h^n(t)$  the  $n$ -th superposition of the formal power series  $h(t)$ .

**Proposition 3.1.** *For any  $n \in \mathbb{N}$ ,  $h^n(t) \equiv t E(n, S) \bmod S^p \mathfrak{m}(\mathcal{K})$*

*Proof.* If  $n = 1$  there is nothing to prove. Suppose proposition is proved for some  $n \in \mathbb{N}$ . Then

$$h^{n+1}(t) = h^n(h(t)) \equiv t E(1, S) E(n, S(h(t))) \bmod \mathfrak{m}(\mathcal{K}) S(h(t))^p.$$

Recall, cf. Subsection 2.2, that  $S = \sum_{l \geq 1} \gamma_{l1} t^l$ , where  $\gamma_{l1} \in W_M(k)$  and  $\gamma_{l1} l = 0$ . Let  $l = l' p^a$  with  $\gcd(l', p) = 1$ . Then  $\gamma_{l1} \in p^{M-a} W_M(k)$ .

With the above notation we have in  $W_M(k)[[t]]$ ,

$$E(1, S)^l = \exp(p^a S + \cdots + p S^{p^{a-1}})^{l'} E(1, S^{p^a})^{l'} \equiv 1 \bmod (p^a, S^p).$$

Therefore (use that  $\gamma_{l1} p^a = 0$ ),

$$S(h(t)) \equiv S(t E(1, S)) \equiv \sum_l \gamma_{l1} t^l E(1, S)^l \equiv \sum_l \gamma_{l1} t^l = S \bmod S^p,$$

and  $h^{n+1}(t) \equiv t E(1, S) E(n, S) \equiv t E(n+1, S) \bmod \mathfrak{m}(\mathcal{K}) S^p$  (use that  $S(h(t))^p \equiv 0 \bmod S^p$ ).  $\square$

**3.2. Specification of lifts  $h_{<p}$ .** Note that  $h(t) = t\alpha^{p^{M-1}}$ , where  $\alpha = E(1, S_0)^p$ , and therefore,  $h$  commutes with  $\sigma$ , cf. Remark in Subsection 1.1. Now suppose that  $h_{<p,0} \in \text{Aut } \mathcal{K}_{<p}$  is a lift of  $h_0$ . Then Proposition 1.1 provides us with a unique  $h_{<p} \in \text{Aut } O(\mathcal{K}_{<p})$  such that  $h_{<p}|_{O(\mathcal{K})} = h$  and  $h_{<p} \bmod p = h_{<p,0}$ . Therefore, we can work with arbitrary lifts  $h_{<p,0}$  of  $h_0$  by working with the appropriate lifts  $h_{<p}$  of  $h$ . Note that all such lifts  $h_{<p}$  commute with  $\sigma$ .

A lift  $h_{<p}$  of  $h$  can be specified by the formalism of nilpotent Artin-Schreier theory as follows.

Define similarly to [8] the continuous  $W_M(k)$ -linear operators  $\mathcal{R}, \mathcal{S} : \mathcal{L}_{\mathcal{K}} \longrightarrow \mathcal{L}_{\mathcal{K}}$  as follows.

Suppose  $\alpha \in \mathcal{L}_k$ .

For  $n > 0$ , set  $\mathcal{R}(t^n \alpha) = 0$  and  $\mathcal{S}(t^n \alpha) = -\sum_{i \geq 0} \sigma^i(t^n \alpha)$ .

For  $n = 0$ , set  $\mathcal{R}(\alpha) = \alpha_0(\text{id}_{\mathcal{L}} \otimes \text{Tr})(\alpha)$ ,  $\mathcal{S}(\alpha) = \sum_{0 \leq j < i < N_0} \sigma^j \alpha_0 \sigma^i \alpha$ , where  $\text{Tr} : W_M(k) \longrightarrow W_M(k)$  is induced by the trace map in  $k/\mathbb{F}_p$  and  $\alpha_0 \in W_M(k)$  with  $\text{Tr} \alpha_0 = 1$  was fixed in Subsection 1.4.

For  $n = -n_1 p^m$ ,  $\gcd(n_1, p) = 1$ , set  $\mathcal{R}(t^n \alpha) = t^{-n_1} \sigma^{-m} \alpha$  and  $\mathcal{S}(t^n \alpha) = \sum_{1 \leq i \leq m} \sigma^{-i}(t^n \alpha)$ .

Similarly to [8] we have the following lemma. (We use also the special case  $\mathfrak{M} = \mathbb{Z}_p$  of Lemma 2.7.)

**Lemma 3.2.** *For any  $b \in \mathcal{L}_{\mathcal{K}}$ ,*

- a)  $b = \mathcal{R}(b) + (\sigma - \text{id}_{\mathcal{L}_{\mathcal{K}}})\mathcal{S}(b)$ ;
- b) *if  $b = b_1 + \sigma c - c$ , where  $b_1 \in \sum_{a \in \mathbb{Z}^+(p)} t^{-a} \mathcal{L}_k + \alpha_0 \mathcal{L}$  and  $c \in \mathcal{L}_{\mathcal{K}}$  then  $\mathcal{R}(b) = b_1$  and  $c - \mathcal{S}(b) \in \mathcal{L}$ ;*
- c) *for any  $n \geq 0$ ,  $\mathcal{R}$  and  $\mathcal{S}$  map  $S^{-n} \mathcal{L}_{\mathfrak{m}(\mathcal{K})}$  to itself.*

According to Subsection 1.6, for the lift  $h_{<p} \in \text{Aut } O(\mathcal{K}_{<p})$  of  $h$  (which is attached to the lift  $h_{<p,0}$  of  $h_0$ ), we have that

$$h_{<p}(f) = c \circ (A \otimes \text{id}_{O(\mathcal{K}_{<p})})f.$$

Here  $c \in \mathcal{L}_{\mathcal{K}}$  and  $A = \text{Ad } h_{<p} \in \text{Aut } \mathcal{L}$  (cf. Subsection 1.6 for the definition of  $\text{Ad } h_{<p}$ ). Similarly to [8] it can be proved that the correspondence  $h_{<p} \mapsto (c, A)$  is a bijection between the set of all lifts  $h_{<p}$  of  $h$  and all  $(c, A) \in \mathcal{L}_{\mathcal{K}} \times \text{Aut } \mathcal{L}$  such that

$$(3.1) \quad (\text{id}_{\mathcal{L}} \otimes h)(e) \circ c = (\sigma c) \circ (A \otimes \text{id}_{O(\mathcal{K})})(e).$$

This allows us to specify a choice of  $h_{<p}$  step by step proceeding from  $h_{<p} \bmod C_s(\mathcal{L}_{\mathcal{K}_{<p}})$  to  $h_{<p} \bmod C_{s+1}(\mathcal{L}_{\mathcal{K}_{<p}})$  where  $1 \leq s < p$ , as follows.

Suppose  $c$  and  $A$  are already chosen modulo  $s$ -th commutators, i.e. we chose  $(c_s, A_s) \in \mathcal{L}_{\mathcal{K}} \times \text{Aut } \mathcal{L}$  satisfying the relation (3.1) modulo  $C_s(\mathcal{L}_{\mathcal{K}})$ .

Then set  $c_{s+1} = c_s + X$  and  $A_{s+1} = A_s + \mathcal{A}$ , where  $X \in C_s(\mathcal{L}_K)$  and  $\mathcal{A} \in \text{Hom}(\mathcal{L}, C_s(\mathcal{L}))$ . Then (3.1) implies that (here  $\mathcal{A}_k = \mathcal{A} \otimes W_M(k)$ )

$$\sigma X - X + \sum_{a \in \mathbb{Z}^0(p)} t^{-a} \mathcal{A}_k(D_{a0}) \equiv$$

$$(3.2) \quad (\text{id}_{\mathcal{L}} \otimes h)e \circ c_s - \sigma c_s \circ (A_s \otimes \text{id}_{O(K)})e \bmod C_{s+1}(\mathcal{L}_K)$$

Now we can specify  $c_{s+1}$  and  $A_{s+1}$  by setting  $X = \mathcal{S}(B_s)$  and  $\sum_{a \in \mathbb{Z}^0(p)} t^{-a} \mathcal{A}_k(D_{a0}) = \mathcal{R}(B_s)$ , where  $B_s$  is the right-hand side of the above recurrent relation. Note that the knowledge of all  $\mathcal{A}_k(D_{a0})$  recovers uniquely the values of  $\mathcal{A}$  on generators of  $\mathcal{L}$  and gives well-defined  $A_{s+1} \in \text{Aut } \mathcal{L}$ . Clearly,  $(c_{s+1}, A_{s+1})$  satisfies the relation (3.1) modulo  $C_{s+1}(\mathcal{L}_K)$ . Finally, we obtain the solution  $(c^0, A^0) := (c_p, A_p)$  of (3.1) and can use it to specify uniquely the lift  $h_{<p}^0$  of  $h$ .

**3.3. The group  $\tilde{\mathcal{G}}_h$ .** Consider the group of all continuous automorphisms of  $\mathcal{K}_{<p}$  such that their restriction to  $\mathcal{K}$  belongs to the closed subgroup in  $\text{Aut } \mathcal{K}$  generated by  $h_0$ . These automorphisms admit unique lifts to automorphisms of  $O(\mathcal{K}_{<p})$  such that their restriction to  $O(\mathcal{K})$  belongs to the subgroup  $\langle h \rangle$  of  $\text{Aut } O(\mathcal{K})$  generated by  $h$ , cf. the beginning of Subsection 3.2. Denote the group of these lifts by  $\tilde{\mathcal{G}}_h$ .

Use the identification  $\eta_M$  from Subsection 1.4 to obtain a natural short exact sequence of profinite  $p$ -groups

$$(3.3) \quad 1 \longrightarrow G(\mathcal{L}) \longrightarrow \tilde{\mathcal{G}}_h \longrightarrow \langle h \rangle \longrightarrow 1$$

For any  $s \geq 2$ , the  $s$ -th commutator subgroup  $C_s(\tilde{\mathcal{G}}_h)$  is a normal subgroup in  $G(\mathcal{L})$ . Therefore,  $\mathcal{L}_h(s) := C_s(\tilde{\mathcal{G}}_h)$  is a Lie subalgebra of  $\mathcal{L}$ . Set  $\mathcal{L}_h(1) = \mathcal{L}$ . Clearly, for any  $s_1, s_2 \geq 1$ ,  $[\mathcal{L}_h(s_1), \mathcal{L}_h(s_2)] \subset \mathcal{L}_h(s_1 + s_2)$ , in other words, the filtration  $\{\mathcal{L}_h(s)\}_{s \geq 1}$  is central.

**Theorem 3.3.** *For all  $s \in \mathbb{N}$ ,  $\mathcal{L}_h(s) = \mathcal{L}(s)$ .*

*Proof.* Use the notation from Subsection 2.5. Obviously, we have:

- $\mathcal{L}(s+1) = \left( \mathcal{K}^* / \mathcal{K}^{*p^M} \right)^{(s+1)} + \mathcal{L}(s+1) \cap C_2(\mathcal{L})$ , where the  $W_M(k)$ -module  $\left( \mathcal{K}^* / \mathcal{K}^{*p^M} \right)^{(s+1)}$  is generated by all  $V_{(b,m)}$  with  $m \geq s+1$  (for the definition of  $V_{(b,m)}$  cf. Proposition 2.6) and  $\mathcal{L}(s+1) \cap C_2(\mathcal{L}) = \sum_{s_1+s_2=s+1} [\mathcal{L}(s_1), \mathcal{L}(s_2)]$ ;

- $\mathcal{L}_h(s+1)$  is the ideal in  $\mathcal{L}$  generated by  $[\mathcal{L}_h(s), \mathcal{L}]$  and all elements of the form  $(\text{Ad} h_{<p})l \circ (-l)$ , where  $l \in \mathcal{L}_h(s)$  and  $h_{<p}$  is a lift of  $h$ .

Consider the elements  $V_{(0)}$  and  $V_{(b,m),i}$  introduced in the end of Section 2). Recall that  $m \in \mathbb{N}$ ,  $1 \leq b < e^*$  and  $\gcd(b, p) = 1$ .



**Lemma 3.4.** *There is a lift  $h_{<p}^1$  such that if  $(\text{Ad}h_{<p}^1)V_{(0)} = \tilde{V}_{(0)}$  and for all  $b, m, i$ ,  $(\text{Ad}h_{<p}^1)V_{(b,m),i} = \tilde{V}_{(b,m),i}$  then*

- a)  $\tilde{V}_{(0)} \equiv V_{(0)} \pmod{C_2(\mathcal{L})}$ ;
- b)  $\tilde{V}_{(b,m),i} \equiv V_{(b,m),i} + bV_{(b,m+1),i} \pmod{(\mathcal{L}(m+2) + \mathcal{L}(m+1) \cap C_2(\mathcal{L}))}$ .

We shall prove this Lemma below.

Note the following immediate applications of this lemma:

- (a) if  $l \in \mathcal{L}(s)$  then  $(\text{Ad}h_{<p}^1)l \circ (-l) \in \mathcal{L}(s+1)$ ;
- (b) if  $l \in \left(\mathcal{K}^*/\mathcal{K}^{*p^M}\right)^{(s+1)}$  then there is an  $l' \in \left(\mathcal{K}^*/\mathcal{K}^{*p^M}\right)^{(s)}$  such that  $(\text{Ad}h_{<p}^1)l' \circ (-l') \equiv l \pmod{\mathcal{L}(s+1) \cap C_2(\mathcal{L})}$ .

Now we can finish the proof of our theorem.

Clearly,  $\mathcal{L}_h(1) = \mathcal{L}(1)$ .

Suppose  $s_0 \geq 1$  and for  $1 \leq s \leq s_0$ , we have  $\mathcal{L}_h(s) = \mathcal{L}(s)$ .

Then  $[\mathcal{L}_h(s_0), \mathcal{L}] = [\mathcal{L}(s_0), \mathcal{L}(1)] \subset \mathcal{L}(s_0+1)$  and applying (a) we obtain that  $\mathcal{L}_h(s_0+1) \subset \mathcal{L}(s_0+1)$ .

In the opposite direction, note that by inductive assumption,

$$\mathcal{L}(s_0+1) \cap C_2(\mathcal{L}) = \sum_{s_1+s_2=s_0+1} [\mathcal{L}_h(s_1), \mathcal{L}_h(s_2)] \subset \mathcal{L}_h(s_0+1)$$

and then from (b) we obtain that  $\left(\mathcal{K}^*/\mathcal{K}^{*p^M}\right)^{(s_0+1)} \subset \mathcal{L}_h(s_0+1)$ . So,  $\mathcal{L}(s_0+1) \subset \mathcal{L}_h(s_0+1)$ . The theorem is completely proved.  $\square$

*Proof of Lemma 3.4.* Let

$$\tilde{e}^\dagger := (\text{Ad}h_{<p}^1 \otimes \text{id}_{O(\mathcal{K})})e^\dagger = \sum_{i,b,m} \frac{t^b}{s_m} \beta_i \tilde{V}_{(b,m),i} + \alpha_{(0)} \tilde{V}_{(0)}.$$

Similarly to Subsection 3.2 there is  $c^1 \in \mathcal{L}_{\mathcal{K}}$  such that

$$(3.4) \quad (\text{id}_{\mathcal{L}} \otimes h)e^\dagger \circ c^1 = (\sigma c^1) \circ \tilde{e}^\dagger.$$

and the choice of  $h_{<p}^1$  can be specified by an analog of the recurrent procedure from the end of Subsection 3.2.

Namely, set  $c_1^1 = 0$  and  $A_1^1 = \text{id}_{\mathcal{L}}$ . Then for  $1 \leq s < p$ ,  $(c_{s+1}^1, A_{s+1}^1)$  can be defined as follows:

- $B_s = (\text{id}_{\mathcal{L}} \otimes h)e^\dagger \circ c_s^1 - (\sigma c_s^1) \circ (A_s^1 \otimes \text{id}_{\mathcal{K}})e^\dagger$
- $X_s = \mathcal{S}(B_s)$ ,  $(\mathcal{A}_s \otimes \text{id}_{\mathcal{K}})e^\dagger = \mathcal{R}(B_s)$ ;
- $c_{s+1}^1 = c_s^1 + X_s$ ,  $A_{s+1}^1 = A_s^1 + \mathcal{A}_s$

This gives the system of compatible on  $1 \leq s \leq p$  solutions  $(c_s^1, A_s^1) \in \mathcal{L}_{\mathcal{K}} \times \text{Aut } \mathcal{L}$  of (3.4) modulo  $C_s(\mathcal{L}_{\mathcal{K}})$  and  $(c^1, A^1) := (c_p^1, A_p^1)$  defines  $h_{<p}^1$ .

Let

$$\tilde{\mathcal{N}}^{(2)} := \sum_{i \geq 2} S^{-i}(\mathcal{L}(i) \cap C_2(\mathcal{L}))_{\mathfrak{m}(\mathcal{K})} \subset \mathcal{N}^{(2)}.$$

Note that  $[\mathcal{N}, \mathcal{N}] \subset \tilde{\mathcal{N}}^{(2)}$ . Consider the following properties.

a)  $(\text{id}_{\mathcal{L}} \otimes h)(e^\dagger) = e^\dagger + e_1^+ + e_1^- \bmod S^2\mathcal{N}$ , where  $e_1^+, e_1^- \in S\mathcal{N}$  and

$$e_1^- = \sum_{i,b,m} \frac{bt^b}{S_m} \beta_i V_{(b,m+1),i}, \quad e_1^+ = \sum_{b,i} bt^b \beta_i V_{(b,1),i}$$

(use that  $h(S) \equiv S(h(t)) \equiv S \bmod S^p$ , cf. the proof of Proposition 3.1).

b)  $\tilde{e}^\dagger \equiv e^\dagger \bmod S\mathcal{N}$  and  $c^1 \in S\mathcal{N}$  (use that for all  $s$ ,  $B_s \in S\mathcal{N}$  and  $\mathcal{R}$  and  $\mathcal{S}$  map  $S\mathcal{N}$  to itself).

c)  $(-\sigma c^1) \circ (\text{id}_{\mathcal{L}} \otimes h)(e^\dagger) \circ c^1 \equiv (c^1 - \sigma c^1) + e^\dagger + e_1^\dagger \bmod S^2\mathcal{N} + S\tilde{\mathcal{N}}^{(2)}$   
(use that  $c \in S\mathcal{N}$  and  $(\text{id}_{\mathcal{L}} \otimes h)(e^\dagger) \in \mathcal{N}$ )

d) Apply  $\mathcal{R}$  to the congruence from c), use that  $S^2\mathcal{N} + S\tilde{\mathcal{N}}^{(2)}$  is mapped by  $\mathcal{R}$  to itself and  $\mathcal{R}(c^1 - \sigma c^1) = \mathcal{R}(e_1^+) = 0$

$$\tilde{e}^\dagger \equiv \sum_{i,b,m} \frac{t^b}{S_m} \beta_i (V_{(b,m),i} + bV_{(b,m+1),i}) + \alpha_0 V_{(0)} \bmod S^2\mathcal{N} + S\tilde{\mathcal{N}}^{(2)}.$$

It remains to note that the last congruence is equivalent to the statement of our lemma.  $\square$

**3.4. The group  $\mathcal{G}_h$ .** Let  $\mathcal{G}_h = \tilde{\mathcal{G}}_h / \tilde{\mathcal{G}}_h^{p^M} C_p(\tilde{\mathcal{G}}_h)$ .

**Proposition 3.5.** *Exact sequence (3.3) induces the following exact sequence of  $p$ -groups*

$$(3.5) \quad 1 \longrightarrow G(\mathcal{L})/G(\mathcal{L}(p)) \longrightarrow \mathcal{G}_h \longrightarrow \langle h \rangle \bmod \langle h^{p^M} \rangle \longrightarrow 1$$

*Proof.* Set

$$\mathcal{M} := \mathcal{N} + \mathcal{L}(p)_{\mathcal{K}} = \sum_{1 \leq s < p} S^{-s} \mathcal{L}(s)_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(p)_{\mathcal{K}}$$

$$\mathcal{M}_{<p} := \sum_{1 \leq s < p} S^{-s} \mathcal{L}(s)_{\mathfrak{m}(\mathcal{K}_{<p})} + \mathcal{L}(p)_{\mathcal{K}_{<p}}$$

where  $\mathfrak{m}(\mathcal{K}_{<p}) = W_M(\mathfrak{m}_{<p}) \cap O(\mathcal{K}_{<p})$  and  $\mathfrak{m}_{<p}$  is the maximal ideal of the valuation ring of  $\mathcal{K}_{<p}$ .

Then  $\mathcal{M}$  has the induced structure of Lie  $W_M(k)$ -algebra (use the Lie bracket from  $\mathcal{L}_{\mathcal{K}}$ ) and  $S^{p-1}\mathcal{M}$  is an ideal in  $\mathcal{M}$ . Similarly,  $\mathcal{M}_{<p}$  is a Lie  $W_M(k)$ -algebra (containing  $\mathcal{M}$  as its subalgebra) and  $S^{p-1}\mathcal{M}_{<p}$  is an ideal in  $\mathcal{M}_{<p}$ . Note that  $e \in \mathcal{M}$ ,  $f \in \mathcal{M}_{<p}$ ,  $S^{p-1}\mathcal{M}_{<p} \cap \mathcal{M} = S^{p-1}\mathcal{M}$ , and we have a natural embedding of  $\bar{\mathcal{M}} := \mathcal{M}/S^{p-1}\mathcal{M}$  into  $\bar{\mathcal{M}}_{<p} := \mathcal{M}_{<p}/S^{p-1}\mathcal{M}_{<p}$ . For  $i \geq 0$ , we have also  $(\text{id}_{\mathcal{L}} \otimes h - \text{id}_{\mathcal{M}})^i \mathcal{M} \subset S^i \mathcal{M}$ .

Consider the orbit of  $\bar{f} := f \bmod S^{p-1}\mathcal{M}_{<p}$  with respect to the natural action of  $\tilde{\mathcal{G}}_h \subset \text{Aut } O(\mathcal{K}_{<p})$  on  $\tilde{\mathcal{M}}_{<p}$ . Prove that the stabilizer  $\mathcal{H}$  of  $\bar{f}$  equals  $\tilde{\mathcal{G}}_h^{p^M} C_p(\tilde{\mathcal{G}}_h)$ .

If  $l \in G(\mathcal{L})$  then  $\eta_M^{-1}(l) \in \mathcal{G}_{<p}$  sends  $f$  to  $f \circ l$ . This means that for  $l \in \mathcal{L} \cap \mathcal{H}$  we have

$$l \in S^{p-1}\mathcal{M}_{<p} \cap \mathcal{L} = S^{p-1}\mathcal{M} \cap \mathcal{L} = \mathcal{L}(p)_\mathcal{K} \cap \mathcal{L} = \mathcal{L}(p) = C_p(\tilde{\mathcal{G}}_h).$$

Therefore,  $\mathcal{H} \cap G(\mathcal{L}) = C_p(\tilde{\mathcal{G}}_h) \subset \mathcal{H}$  and we obtain the embedding

$$\kappa : G(\mathcal{L})/G(\mathcal{L}(p)) \longrightarrow \tilde{\mathcal{G}}_h/\mathcal{H}.$$

Now consider the lift  $h_{<p}^0$  from the end of Subsection 3.2.

Note that  $\tilde{\mathcal{G}}_h^{p^M} \bmod C_p(\tilde{\mathcal{G}}_h)$  is generated by  $h_{<p}^{0p^M}$ . Indeed, any finite  $p$ -group of nilpotent class  $< p$  is  $P$ -regular, cf. [10] Subsection 12.3. In particular, for any  $g \in G(\mathcal{L})$ ,  $(h_{<p}^0 \circ g)^{p^M} \equiv h_{<p}^{0p^M} \circ g' \bmod C_p(\tilde{\mathcal{G}}_h)$ , where  $g'$  is the product of  $p^M$ -th powers of elements from  $G(\mathcal{L})$ , but  $G(\mathcal{L})$  has period  $p^M$ .

As earlier,  $h_{<p}^0 f = c^0 \circ (A^0 \otimes \text{id}_\mathcal{K})f$ . Note that  $c^0 \in S\mathcal{M}$  (proceed similarly to the proof of Lemma 3.4, step b)).

Then  $h_{<p}^{0p^M}(f) =$

$$(\text{id} \otimes h)^{p^M-1} \left( c^0 \circ (A^0 \otimes h^{-1})c^0 \circ \dots \circ (A^0 \otimes h^{-1})^{p^M-1}c^0 \right) \circ (A^{0p^M} \otimes \text{id})f.$$

Clearly,  $(A^0 - \text{id}_\mathcal{L})^p \mathcal{L} \subset \mathcal{L}(p)$  and, therefore,  $(A^{0p^M} \otimes \text{id})\bar{f} = \bar{f}$ .

Similarly,  $B = A^0 \otimes h^{-1}$  is an automorphism of the Lie algebra  $\mathcal{M}$ , and for all  $s \geq 0$ ,  $(B - \text{id}_\mathcal{M})(S^s \mathcal{M}) \subset S^{s+1} \mathcal{M}$ .

**Lemma 3.6.** *For any  $m \in S\mathcal{M}$ ,  $m \circ B(m) \circ \dots \circ B^{p^M-1}m \in S^p \mathcal{M}$ .*

*Proof.* Consider the Lie algebra  $\mathfrak{M} = S\mathcal{M}/S^p \mathcal{M}$  with the filtration  $\{\mathfrak{M}(i)\}_{i \geq 1}$  induced by the filtration  $\{S^i \mathcal{M}\}_{i \geq 1}$ . This filtration is central, i.e. for any  $i, j \geq 1$ ,  $[\mathfrak{M}(i), \mathfrak{M}(j)] \subset \mathfrak{M}(i+j)$ . In particular, the nilpotent class of  $\mathfrak{M}$  is  $< p$ .

The operator  $B$  induces the operator on  $\mathfrak{M}$  which we denote also by  $B$ . Clearly,  $B = \widetilde{\exp} \mathcal{B}$  where  $\mathcal{B}$  is a differentiation on  $\mathfrak{M}$  such that for all  $i \geq 1$ ,  $\mathcal{B}(\mathfrak{M}(i)) \subset \mathfrak{M}(i+1)$ .

Let  $\widetilde{\mathfrak{M}}$  be a semi-direct product of  $\mathfrak{M}$  and the trivial Lie algebra  $(\mathbb{Z}/p^M)w$  via  $\mathcal{B}$ . This means that  $\widetilde{\mathfrak{M}} = \mathfrak{M} \oplus (\mathbb{Z}/p^M)w$  as  $\mathbb{Z}/p^M$ -module,  $\mathfrak{M}$  and  $(\mathbb{Z}/p^M)w$  are Lie subalgebras of  $\widetilde{\mathfrak{M}}$  and for any  $m \in \mathfrak{M}$ ,  $[m, w] = \mathcal{B}(m)$ . Clearly,  $C_2(\widetilde{\mathfrak{M}}) = [\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{M}}] \subset \mathfrak{M}(2)$ . This implies that  $\widetilde{\mathfrak{M}}$  has nilpotent class  $< p$  and we can consider the  $p$ -group  $G(\widetilde{\mathfrak{M}})$ . This group has nilpotent class  $< p$  and period  $p^M$  (because for any  $\bar{m} \in \widetilde{\mathfrak{M}}$ , its  $p^M$ -th power in  $G(\widetilde{\mathfrak{M}})$  equals  $p^M \bar{m} = 0$ ).

Note that the conjugation by  $w$  in  $G(\widetilde{\mathfrak{M}})$  is given by the automorphism  $\widetilde{\exp} \mathcal{B} = B$ . Indeed, if  $m \in \mathfrak{M}$  then

$$B(m) = (\widetilde{\exp} \mathcal{B})m = \sum_{0 \leq n < p} \mathcal{B}^n(m)/n! = (-w) \circ m \circ w$$

(use very well-known formula in a free associative algebra  $\mathbb{Q}[[X, Y]]$ ,

$$\exp(-Y) \exp(X) \exp(Y) = \exp(X + \dots + (\text{ad}^n Y)X/n! + \dots),$$

where  $\text{ad} Y : X \mapsto [X, Y]$ ).

In particular, for any element  $\bar{m} = m \bmod \mathcal{N}(p) \in \mathfrak{M}$ , we have  $w_1 \circ \bar{m} = B(\bar{m}) \circ w_1$ , where  $w_1 = -w$ . Therefore,  $0 = (\bar{m} \circ w_1)^{p^M} = \bar{m} \circ B(\bar{m}) \circ \dots \circ B^{p^M-1}(\bar{m}) \circ w_1^{p^M}$ , and it remains to note that  $w_1^{p^M} = 0$ .  $\square$

Applying the above Lemma we obtain that

$$c^0 \circ (A^0 \otimes h^{-1})c^0 \circ \dots \circ (A^0 \otimes h^{-1})^{p^M-1}c^0 \in \mathcal{N}(p) \subset S^{p-1}\mathcal{M}$$

and, therefore,  $h_{<p}^{0p^M}(\bar{f}) = 0$ .

Thus, we proved that  $\widetilde{\mathcal{G}}_h^{p^M} C_p(\widetilde{\mathcal{G}}_h) \subset \mathcal{H}$ .

Suppose  $g = h_{<p}^m l \in \mathcal{H}$  with some  $l \in G(\mathcal{L})$ . Then  $g(f) = b \circ f$  where  $b \in S^{p-1}\mathcal{M}_{<p}$ . Note that  $\sigma(b) \in S^{p-1}\mathcal{M}_{<p}$ . Then

$$g(e) \circ b \circ f = g(e) \circ g(f) = g(\sigma f) = \sigma b \circ \sigma f = \sigma b \circ e \circ f$$

implies that  $g(e) \equiv e \bmod S^{p-1}\mathcal{M}$ . Thus  $(\text{id} \otimes h)^m(e) \equiv e \bmod S^{p-1}\mathcal{M}$ .

Now use that  $e \equiv e^\dagger \bmod \mathcal{L}_{\mathfrak{m}(\mathcal{K})} + C_2(\mathcal{L})_{\mathcal{K}}$ , cf. the beginning of the proof of Proposition 2.6.

Clearly,  $\mathcal{L}_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(p)_{\mathcal{K}} \supset S^{p-1}\mathcal{M}$  and, therefore, for the element

$$e_{<p}^\dagger := \sum_{i,b} \sum_{1 \leq m < p} \frac{t^b}{S^m} \beta_i V_{(b,m),i}$$

we obtain  $(\text{id}_{\mathcal{L}} \otimes h)^m(e_{<p}^\dagger) \equiv e_{<p}^\dagger \bmod \mathcal{L}_{\mathfrak{m}(\mathcal{K})} + C_2(\mathcal{L})_{\mathcal{K}}$ . But

$$h^m(e_{<p}^\dagger) \equiv \sum_{i,b} \sum_{1 \leq m < p} \frac{t^b E(bm, S)}{S^m} \beta_i V_{(b,m),i} \bmod \mathcal{L}_{\mathfrak{m}(\mathcal{K})} + \mathcal{L}(p)_{\mathcal{K}}$$

Now following the coefficients for  $V_{(b,p-2),i}$  we obtain  $m \equiv 0 \bmod p^M$ .

Therefore,  $l \in \mathcal{H} \cap G(\mathcal{L}) = C_p(\widetilde{\mathcal{G}}_h)$  and  $\mathcal{H} \subset \widetilde{\mathcal{G}}_h^{p^M} C_p(\widetilde{\mathcal{G}}_h)$ .

Finally, we have  $\widetilde{\mathcal{G}}_h/\mathcal{H} = \mathcal{G}_h$ ,  $\mathcal{H} \bmod C_p(\widetilde{\mathcal{G}}_h) = \langle h_{<p}^{p^M} \rangle$  and, therefore,  $\text{Coker } \kappa = \langle h \rangle \bmod \langle h^{p^M} \rangle$ .  $\square$

**Corollary 3.7.** *If  $L_h$  is a Lie  $\mathbb{Z}/p^M$  algebra such that  $\mathcal{G}_h = G(L_h)$  then (3.5) induces the following short exact sequence of Lie  $\mathbb{Z}/p^M$ -algebras*

$$0 \longrightarrow \mathcal{L}/\mathcal{L}(p) \longrightarrow L_h \longrightarrow (\mathbb{Z}/p^M)h \longrightarrow 0$$

**Remark.** In [8] we studied the structure of the above Lie algebra  $L_h$  in the case  $M = 1$ . The case of arbitrary  $M$  will be considered in a forthcoming paper.

**3.5. Ramification estimates.** Use the identification from Subsection 1.3,  $\eta_M : \text{Gal}(\mathcal{K}_{<p}/\mathcal{K}) = \mathcal{G}_{<p} \simeq G(\mathcal{L})$  and set for all for  $s \in \mathbb{N}$ ,  $\mathcal{K}[s, M] := \mathcal{K}_{<p}^{G(\mathcal{L}^{(s+1)})}$ . Denote by  $v(s, M)$  the maximal upper ramification number of the extension  $\mathcal{K}[s, M]/\mathcal{K}$ . In other words,

$$v(s, M) = \max\{v \mid \mathcal{G}_{<p}^{(v)} \text{ acts non-trivially on } \mathcal{K}[s, M]\}.$$

**Proposition 3.8.** *For all  $s \in \mathbb{N}$ ,  $v(s, M) = p^{M-1}(e^*s - 1)$  (for the definition of  $e^*$  cf, Subsection 2.1).*

*Proof.* Recall, cf. Subsection 1.7, that for any  $v \geq 0$ , the ramification subgroups  $\mathcal{G}_{<p}^{(v)}$  are identified with the ideals  $\mathcal{L}^{(v)}$  of  $\mathcal{L}$ , and for sufficiently large  $N = N(v)$ , the ideal  $\mathcal{L}_k^{(v)}$  is generated by all  $\sigma^n \mathcal{F}_{\gamma, -N}^0$ , where  $\gamma \geq v$ ,  $n \in \mathbb{Z}/N_0$  and the elements  $\mathcal{F}_{\gamma, -N}^0$  are given by (1.3).

Let  $e^0 = e^*(1 - 1/p)$ .

**Lemma 3.9.** *If  $a \in \mathbb{Z}^+(p)$ ,  $u \in \mathbb{N}$  and  $0 \leq c < M$  then the following two conditions are equivalent:*

- a)  $t^a S^{-u} \in \mathfrak{m}(\mathcal{K}) \bmod p^c O(\mathcal{K})$ ;
- b)  $a > e^*u + e^0(c - 1)$ .

*Proof of lemma.* Proposition 2.1c) implies that

$$t^a S^{-u} = t^{a-ue^*} \eta_0 \left( 1 + \sum_{i \geq 1} t^{-ie^0} \eta_i(u) p^i \right)$$

where  $\eta_0$  and all  $\eta_i(u)$  are invertible elements of  $W_M(k)[[t]] \subset O(\mathcal{K})$ . Therefore,  $t^a S^{-u} \in \mathfrak{m}(\mathcal{K}) \bmod p^c O(\mathcal{K})$  if and only if for all  $1 \leq i < c$ ,  $t^{a-ue^*-ie^0} \in \mathfrak{m}(\mathcal{K})$ , i.e.  $a - ue^* - (c-1)e^0 > 0$ . The lemma is proved.  $\square$

**Corollary 3.10.**  *$D_{an} \in \mathcal{L}(u)_k \bmod p^c O(\mathcal{K})$  if and only if we have that  $a \geq e^*(u - 1) + (c - 1)e^0 + 1$ .*

**Lemma 3.11.** *Suppose  $N \geq 0$ .*

- a) *If  $\gamma > p^{M-1}(e^*s - 1)$  then  $\mathcal{F}_{\gamma, -N}^0 \in \mathcal{L}(s + 1)_k$ ;*
- b) *if  $\gamma = p^{M-1}(e^*s - 1)$  then*

$$\mathcal{F}_{\gamma, -N}^0 \equiv p^{M-1} D_{e^*s-1, M-1} \bmod \mathcal{L}(s + 1)_k.$$

*Proof of lemma.* For any  $\gamma > 0$ ,  $\mathcal{F}_{\gamma, -N}^0$  is a  $\mathbb{Z}/p^M$ -linear combination of the monomials of the form

$$X(b; a_1, \dots, a_r; m_2, \dots, m_r) = p^b a_1 [\dots [D_{a_1, b-m_1}, D_{a_2, b-m_2}], \dots, D_{a_r, b-m_r}],$$

where  $0 \leq b < M$ ,  $1 \leq r < p$ , all  $a_i \in \mathbb{Z}^0(p)$ ,  $0 = m_1 \leq m_2 \leq \dots \leq m_r$ , and

$$p^b \left( a_1 + \frac{a_2}{p^{m_2}} + \dots + \frac{a_r}{p^{m_r}} \right) = \gamma.$$

For  $1 \leq i \leq r$ , let  $u_i \in \mathbb{Z}$  be such that (note that  $p^M | e^*$ ,  $p^{M-1} | e^0$  and if  $M = 1$  then  $M - b - 1 = 0$ )

$$1 + e^*(u_i - 1) + e^0(M - b - 1) \leq a_i < e^*u_i + e^0(M - b - 1).$$

This means that all  $D_{a_i, b-m_i} \in \mathcal{L}(u_i)_k \bmod p^{M-b} \mathcal{L}_k$ .

Suppose  $X(b; a_1, \dots, a_r; m_2, \dots, m_r) \notin \mathcal{L}(s+1)_k$ . This implies that  $u_1 + \dots + u_r \leq s$  and, therefore,  $a_1 + \dots + a_r \leq e^*s + re^0(M - b - 1) - r$ .

If  $\gamma > p^{M-1}(e^*s - 1)$  then  $a_1 + \dots + a_r > p^{M-b-1}(e^*s - 1)$  and

$$e^*s + re^0(M - b - 1) - r > p^{M-b-1}(e^*s - 1).$$

Set  $c = M - b - 1$ , then  $0 \leq c < M$  and

$$(p^c - 1)(e^*s - 1) \leq r(e^0c - 1).$$

If  $c = 0$  then  $r \leq 0$ , contradiction.

If  $c \geq 1$  then (use that  $r \leq p - 1$  and  $s \geq 1$ )

$$(1 + p + \dots + p^{c-1})(e^* - 1) \leq e^0c - 1.$$

But then  $e^* = e^0(1 + 1/(p-1)) \geq e^0 + 1$  implies that  $1 + p + \dots + p^{c-1} < c$ . This contradiction proves a).

Suppose  $\gamma = p^{M-1}(e^*s - 1)$ . Then the expression for  $\mathcal{F}_{\gamma, -N}^0$  contains the term  $p^{M-1}D_{e^*s-1, M-1}$ . Take (with above notation) any another monomial  $X(b; a_1, \dots, a_r; m_2, \dots, m_r)$  from the expression of  $\mathcal{F}_{\gamma, -N}^0$ . Clearly,  $r \geq 2$ . As earlier, the assumption that this monomial does not belong to  $\mathcal{L}(s+1)_k$  implies that

$$(p^c - 1)(e^*s - 1) \leq r(e^0c - 1) + 1.$$

If  $c = 0$  then  $r \leq 1$ , contradiction.

If  $c \geq 1$  then again use that  $r \leq p - 1$  to obtain

$$(1 + p + \dots + p^{c-1})(e^*s - 1) \leq e^0c - 1 + 1/(p-1) < e^0c$$

and note that the left-hand side of this inequality  $> ce^0$  (use that  $e^*s - 1 \geq e^* - 1 \geq e^0$ ). The contradiction. The lemma is completely proved.  $\square$

It remains to note that Lemma 3.11 implies that

$$\max\{v \mid \mathcal{L}^{(v)} \not\subset \mathcal{L}(s+1)\} = p^{M-1}(e^*s - 1).$$

Proposition 3.8 is completely proved.  $\square$

#### 4. APPLICATIONS TO THE MIXED CHARACTERISTIC CASE

Let  $K$  be a finite field extension of  $\mathbb{Q}_p$  with the residue field  $k \simeq \mathbb{F}_{p^{N_0}}$  and the ramification index  $e_K$ . Let  $\pi_0$  be a uniformising element in  $K$ . Denote by  $\bar{K}$  an algebraic closure of  $K$  and set  $\Gamma = \text{Gal}(\bar{K}/K)$ . Assume that  $K$  contains a primitive  $p^M$ -th root of unity  $\zeta_M$ .

4.1. For  $n \in \mathbb{N}$ , choose  $\pi_n \in \bar{K}$  such that  $\pi_n^p = \pi_{n-1}$ . Let  $\tilde{K} = \bigcup_{n \in \mathbb{N}} K(\pi_n)$ ,  $\Gamma_{<p} := \Gamma/\Gamma^{p^M} C_p(\Gamma)$  and  $\tilde{\Gamma} = \text{Gal}(\bar{K}/\tilde{K})$ . Then  $\tilde{\Gamma} \subset \Gamma$  induces a continuous group homomorphism  $i : \tilde{\Gamma} \rightarrow \Gamma_{<p}$ .

We have  $\text{Gal}(K(\pi_M)/K) = \langle \tau_0 \rangle^{\mathbb{Z}/p^M}$ , where  $\tau_0(\pi_M) = \pi_M \zeta_M$ . Let  $j : \Gamma_{<p} \rightarrow \text{Gal}(K(\pi_M)/K)$  be a natural epimorphism.

**Proposition 4.1.** *The following sequence*

$$\tilde{\Gamma} \xrightarrow{i} \Gamma_{<p} \xrightarrow{j} \langle \tau_0 \rangle^{\mathbb{Z}/p^M} \rightarrow 1$$

*is exact.*

*Proof.* For  $n > M$ , let  $\zeta_n \in \bar{K}$  be such that  $\zeta_n^p = \zeta_{n-1}$ .

Consider  $\tilde{K}' = \bigcup_{n \geq M} K(\pi_n, \zeta_n)$ . Then  $\tilde{K}'/K$  is Galois with the Galois group  $\Gamma_{\tilde{K}'/K} = \langle \sigma, \tau \rangle$ . Here for any  $n \geq M$  and some  $s_0 \in \mathbb{Z}$ ,  $\sigma \zeta_n = \zeta_n^{1+p^M s_0}$ ,  $\sigma \pi_n = \pi_n$ ,  $\tau(\zeta_n) = \zeta_n$ ,  $\tau \pi_n = \pi_n \zeta_n$  and  $\sigma^{-1} \tau \sigma = \tau^{(1+p^M s_0)^{-1}}$ .

Therefore,  $\Gamma_{\tilde{K}'/K}^{p^M} = \langle \sigma^{p^M}, \tau^{p^M} \rangle$  and for the subgroup of second commutators we have  $C_2(\Gamma_{\tilde{K}'/K}) \subset \langle \tau^{p^M} \rangle \subset \Gamma_{\tilde{K}'/K}^{p^M}$ . This implies that

$$\Gamma_{\tilde{K}'/K}^{p^M} C_p(\Gamma_{\tilde{K}'/K}) = \langle \sigma^{p^M}, \tau^{p^M} \rangle$$

and for  $\Gamma_{\tilde{K}'/K}(M) := \Gamma_{\tilde{K}'/K} / \Gamma_{\tilde{K}'/K}^{p^M} C_p(\Gamma_{\tilde{K}'/K})$ , we obtain a natural exact sequence

$$\langle \sigma \rangle \rightarrow \Gamma_{\tilde{K}'/K}(M) \rightarrow \langle \tau \rangle \bmod \langle \tau^{p^M} \rangle = \langle \tau_0 \rangle^{\mathbb{Z}/p^M} \rightarrow 1.$$

Note that  $\Gamma_{\tilde{K}'}$  together with a lift  $\hat{\sigma} \in \tilde{\Gamma}$  of  $\sigma$  generate  $\tilde{\Gamma}$ . The above short exact sequence implies that  $\text{Ker} \left( \Gamma_{<p} \rightarrow \langle \tau_0 \rangle^{\mathbb{Z}/p^M} \right)$  is generated by  $\hat{\sigma}$  and the image of  $\Gamma_{\tilde{K}'}$ . So, this kernel coincides with the image of  $\tilde{\Gamma}$  in  $\Gamma_{<p}$ .  $\square$

4.2. Let  $R$  be Fontaine's ring. We have a natural embedding  $k \subset R$  and an element  $t_0 = (\pi_n \bmod p)_{n \geq 0} \in R$ . Then we can identify the field  $k((t_0))$  with the field  $\mathcal{K}$  from Sections 1-3. If  $R_0 = \text{Frac } R$  then  $\mathcal{K}$  is a closed subfield of  $R_0$  and the theory of the field-of-norms functor identifies  $R_0$  with the completion of the separable closure  $\mathcal{K}_{sep}$  of  $\mathcal{K}$  in  $R_0$ . Note that  $R$  is the valuation ring of  $R_0$  and denote by  $\mathfrak{m}_R$  the maximal ideal of  $R$ .

This allows us to identify  $\mathcal{G} = \text{Gal}(\mathcal{K}_{sep}/\mathcal{K})$  with  $\tilde{\Gamma} \subset \Gamma \subset \text{Aut } R_0$ . This identification is compatible with the appropriate ramification filtrations. Namely, if  $\varphi_{\tilde{K}/K}$  is the Herbrand function of the (arithmetically profinite) field extension  $\tilde{K}/K$  then for any  $v \geq 0$ ,  $\mathcal{G}^{(v)} = \Gamma^{(v_1)} \cap \tilde{\Gamma}$ , where  $v_1 = \varphi_{\tilde{K}/K}(v)$ .

Let as earlier,  $\mathcal{G}_{<p} = \mathcal{G}/\mathcal{G}^{p^M} C_p(\mathcal{G})$ . Then the embedding  $\mathcal{G} = \tilde{\Gamma} \subset \Gamma$  induces a natural continuous morphism  $\iota$  of the infinite group  $\mathcal{G}_{<p}$  to the

finite group  $\Gamma_{<p}$ . Therefore, by Proposition 4.1 we obtain the following exact sequence

$$(4.1) \quad \mathcal{G}_{<p} \xrightarrow{\iota} \Gamma_{<p} \xrightarrow{j} \langle \tau_0 \rangle^{\mathbb{Z}/p^M} \longrightarrow 1.$$

Let  $\zeta_M = 1 + \sum_{i \geq 1} [\beta_i] \pi_0^i$  with all  $\beta_i \in k$ . Consider the identification of rings  $R/t_0^{e_K} \simeq O_{\bar{K}}/p$  given by  $(r_0, \dots, r_n, \dots) \mapsto r_0$ . If  $\varepsilon = (\zeta_n)_{n \geq 0}$  is Fontaine's element such that  $\zeta_M$  is our fixed  $p^M$ -th root of unity then we have in  $W_M(R)$  the following congruence (as earlier,  $t = (t_0, \dots, 0) \in W_M(R)$ )

$$(4.2) \quad \sigma^{-M} \varepsilon \equiv 1 + \sum_{i \geq 1} \beta_i t^i \pmod{(t^{e_K}, p)}.$$

Now we can specify the choice of the elements  $S_0, S \in \mathfrak{m}(\mathcal{K})$ , cf. Subsection 2.1, by setting  $E(1, S_0) = 1 + \sum_i \beta_i t^i$  and  $S = [p]^M(S_0)$ . Note that  $S \pmod{p}$  generates the ideal  $(t_0^{e^*})$  in  $O_{\mathcal{K}} = k[[t_0]]$ , where  $e^* = pe_K/(p-1)$ . Now congruence (4.2) can be rewritten in the following form

$$\sigma^{-M} \varepsilon \equiv E(1, S_0) \pmod{(\sigma^{-1} S^{p-1}, p)}.$$

Applying  $\sigma$  we obtain

$$\sigma^{-M+1} \varepsilon \equiv E(1, [p]S_0) \pmod{(S^{p-1}, p)},$$

and then taking  $p^{M-1}$ -th power

$$\varepsilon \equiv E(1, S) \pmod{S^{p-1} W_M(R)}.$$

4.3. Let  $v_{\mathcal{K}}$  be the extension of the normalized valuation on  $\mathcal{K}$  to  $R_0$ . Consider a continuous field embedding  $\eta_0 : \mathcal{K} \rightarrow R_0$  compatible with  $v_{\mathcal{K}}$ . Denote by  $\text{Iso}(\eta_0, \mathcal{K}_{<p}, R_0)$  the set of all extensions  $\eta_{<p,0}$  of  $\eta_0$  to  $\mathcal{K}_{<p}$ . This set is a principal homogeneous space over  $\mathcal{G}_{<p} = G(\mathcal{L})$ .

Choose a lift  $\eta : O(\mathcal{K}) \rightarrow W_M(R_0)$  such that  $\eta \pmod{p} = \eta_0$  and  $\eta\sigma = \sigma\eta$ . Proceeding similarly to Subsection 1.1 we can identify the set of all lifts  $\eta_{0,<p}$  of  $\eta_0$  from  $\text{Iso}(\eta_0, \mathcal{K}_{<p}, R_0)$  with the set of all (commuting with  $\sigma$ ) lifts  $\eta_{<p}$  of  $\eta$  from  $\text{Iso}(\eta, O(\mathcal{K}_{<p}), W_M(R_0))$ .

Specify uniquely each lift  $\eta_{<p}$  by the knowledge of  $\eta_{<p}(f) \in \mathcal{L}_{R_0}$  in the set of all solutions  $f' \in \mathcal{L}_{R_0}$  of the equation  $\sigma f' = \eta(e) \circ f'$ . (The elements  $e \in \mathcal{L}_{\mathcal{K}}$  and  $f \in \mathcal{L}_{\mathcal{K}_{<p}}$  were chosen in Subsection 1.4.)

Consider the appropriate submodules  $\mathcal{M} \subset \mathcal{L}_{\mathcal{K}}$ ,  $\mathcal{M}_{<p} \subset \mathcal{L}_{\mathcal{K}_{<p}}$  from Subsection 3.4 and define similarly

$$\mathcal{M}_{R_0} = \sum_{1 \leq s < p} S^{-s} \mathcal{L}(s)_{\mathfrak{m}(R)} + \mathcal{L}(p)_{R_0} \subset \mathcal{L}_{R_0},$$

where  $\mathfrak{m}(R) = W_M(\mathfrak{m}_R)$ . We know that  $e \in \mathcal{M}$ ,  $f \in \mathcal{M}_{<p}$  and for similar reasons, all  $\eta_{<p}(f) \in \mathcal{M}_{R_0}$ .

**Lemma 4.2.** *With above notation suppose that  $\eta(e) \equiv e \pmod{S^{p-1} \mathcal{M}_{R_0}}$ . Then there is  $c \in S^{p-1} \mathcal{M}_{R_0}$  such that  $\eta(e) = \sigma c \circ e \circ (-c)$ .*



*Proof.* Note that  $S^{p-1}\mathcal{M}_{R_0}$  is an ideal in  $\mathcal{M}_{R_0}$  and for any  $i \in \mathbb{N}$  and  $m \in S^{p-1}C_i(\mathcal{M}_{R_0})$ , there is  $c \in S^{p-1}C_i(\mathcal{M}_{R_0})$  such that  $\sigma c - c = m$ . (Use that  $\sigma$  is topologically nilpotent on  $S^{p-1}C_i(\mathcal{M}_{R_0})$ .)

Therefore, there is  $c_1 \in S^{p-1}\mathcal{M}_{R_0}$  such that  $\eta(e) = e + \sigma c_1 - c_1$ . This implies that  $\eta(e) \circ c_1 \equiv \sigma c_1 \circ e \pmod{S^{p-1}C_2(\mathcal{M}_{R_0})}$ . Similarly, there is  $c_2 \in S^{p-1}C_2(\mathcal{M}_{R_0})$  such that  $\eta(e) \circ c_1 + c_2 = \sigma c_2 + \sigma c_1 \circ e_0$  and  $\eta(e_0) \circ c_1 \circ c_2 \equiv \sigma c_2 \circ \sigma c_1 \circ e_0 \pmod{S^{p-1}C_3(\mathcal{M}_{R_0})}$ , and so on.

After  $p-1$  iterations we obtain for  $1 \leq i < p$  the elements  $c_i \in S^{p-1}C_i(\mathcal{M}_{R_0})$  such that

$$\eta(e) \circ (c_1 \circ \cdots \circ c_{p-1}) = \sigma(c_{p-1} \circ \cdots \circ c_1) \circ e.$$

The lemma is proved.  $\square$

The above lemma implies the following properties:

**Proposition 4.3.** a) If  $\eta(e) \equiv e \pmod{S^{p-1}\mathcal{M}_{R_0}}$  then for any  $\eta_{<p} \in \text{Iso}(\eta, \mathcal{K}_{<p}, R_0)$ , there is a unique  $l \in G(\mathcal{L}) \pmod{G(\mathcal{L}(p))}$  such that

$$\eta_{<p}(f) \equiv f \circ l \pmod{S^{p-1}\mathcal{M}_{R_0}}.$$

b) Suppose  $\eta', \eta'' : O(\mathcal{K}) \rightarrow W_M(R_0)$  are such that

$$\eta'(t) \equiv \eta''(t) \pmod{S^{p-1}W_M(\mathfrak{m}_R)}.$$

If  $\eta'_{<p} \in \text{Iso}(\eta', O(\mathcal{K}_{<p}), W_M(R_0))$  and  $\eta''_{<p} \in \text{Iso}(\eta'', O(\mathcal{K}_{<p}), W_M(R_0))$  then there is a unique  $l \in G(\mathcal{L})$  such that

$$\eta'_{<p}(f) \equiv \eta''_{<p}(f) \circ l \pmod{S^{p-1}\mathcal{M}_{R_0}}.$$

4.4. The action of  $\Gamma = \text{Gal}(\bar{K}/K)$  on  $R_0$  is strict and, therefore, the elements  $g \in \Gamma$  can be identified with all continuous field embeddings  $g : \mathcal{K}_{\text{sep}} \rightarrow R_0$  such that  $g|_{\mathcal{K}}$  belongs to the set  $\langle \tau_0 \rangle = \{\tau_0^a \mid a \in \mathbb{Z}_p\}$ .

Extend  $\tau_0$  now to a continuous embedding  $\tau : O(\mathcal{K}) \rightarrow W_M(R_0)$  uniquely determined by the condition  $\tau(t) = t\varepsilon$ . Clearly,  $\tau$  commutes with  $\sigma$ . Then the results of Subsection 1.1 imply that the elements of  $\Gamma$  are identified with the continuous embeddings  $g : O(\mathcal{K}_{\text{sep}}) \rightarrow W_M(R_0)$  such that  $g|_{O(\mathcal{K})}$  belongs to the set  $\langle \tau \rangle$ .

Consider  $h_0 \in \text{Aut}(\mathcal{K})$  such that  $h_0(t_0) = t_0 E(1, S \pmod{p})$  and  $h_0|_k = \text{id}$ . Then its lift  $h \in \text{Aut}O(\mathcal{K})$  such that  $h(t) = tE(1, S)$  commutes with  $\sigma$  and there are the appropriate groups  $\tilde{\mathcal{G}}_h$  and  $\mathcal{G}_h$  from Section 3.

Clearly,  $h(t) \equiv \tau(t) \pmod{S^{p-1}\mathfrak{m}_R}$  and we can apply Proposition 4.3b). This implies that the  $\Gamma$ -orbit of  $f \pmod{S^{p-1}\mathcal{M}_{R_0}}$  is contained in the  $\tilde{\mathcal{G}}_h$ -orbit of  $f \pmod{S^{p-1}\mathcal{M}_{R_0}}$ . Therefore, there is a map of sets  $\kappa : \Gamma \rightarrow \mathcal{G}_h$  uniquely determined by the requirement that for any  $g \in \Gamma$ ,

$$(\text{id}_{\mathcal{L}} \otimes g)f \equiv (\text{id}_{\mathcal{L}} \otimes \kappa(g))f \pmod{S^{p-1}\mathcal{M}_{R_0}}.$$

(Use that  $\mathcal{G}_h$  strictly acts on the  $\tilde{\mathcal{G}}_h$ -orbit of  $f \pmod{S^{p-1}\mathcal{M}_{R_0}}$ .)

**Proposition 4.4.**  $\kappa$  induces a group isomorphism  $\kappa_{<p} : \Gamma_{<p} \rightarrow \mathcal{G}_h$ .

*Proof.* Suppose  $g_1, g \in \Gamma$ . Let  $c \in \mathcal{L}_K$  and  $A \in \text{Aut } \mathcal{L}$  be such that  $(\text{id}_{\mathcal{L}} \otimes \kappa(g))f = c \circ (A \otimes \text{id}_{\mathcal{K}_{< p}})f$ . Then we have the following congruences modulo  $S^{p-1}\mathcal{M}_{R_0}$

$$\begin{aligned} (\text{id}_{\mathcal{L}} \otimes \kappa(g_1 g))f &\equiv (\text{id}_{\mathcal{L}} \otimes g_1 g)f \equiv (\text{id}_{\mathcal{L}} \otimes g_1)(\text{id}_{\mathcal{L}} \otimes g)f \equiv \\ &(\text{id}_{\mathcal{L}} \otimes g_1)(\text{id}_{\mathcal{L}} \otimes \kappa(g))f \equiv (\text{id}_{\mathcal{L}} \otimes g_1)(c \circ (A \otimes \text{id}_{\mathcal{K}_{< p}})f) \equiv \\ &(\text{id}_{\mathcal{L}} \otimes g_1)c \circ (A \otimes g_1)f \equiv (\text{id}_{\mathcal{L}} \otimes \kappa(g_1))c \circ (A \otimes \kappa(g_1))f \equiv \\ &(\text{id}_{\mathcal{L}} \otimes \kappa(g_1))(c \circ (A \otimes \text{id}_{\mathcal{K}_{< p}})f) \equiv (\text{id}_{\mathcal{L}} \otimes \kappa(g_1))(\text{id}_{\mathcal{L}} \otimes \kappa(g))f \\ &\equiv (\text{id}_{\mathcal{L}} \otimes \kappa(g_1)\kappa(g))f \end{aligned}$$

and, therefore,  $\kappa(g_1 g) = \kappa(g_1)\kappa(g)$  (use that  $\mathcal{G}_h$  acts strictly on the orbit of  $f$ ).

Therefore,  $\kappa$  factors through the natural projection  $\Gamma \rightarrow \Gamma_{< p}$  and defines the group homomorphism  $\kappa_{< p} : \Gamma_{< p} \rightarrow \mathcal{G}_h$ .

Recall that we have the field-of-norms identification  $\tilde{\Gamma} = \mathcal{G}$  and, therefore,  $\kappa_{< p}$  identifies the groups  $\kappa(\tilde{\Gamma})$  and  $G(\mathcal{L}/\mathcal{L}(p)) \subset \mathcal{G}_h$ . Besides,  $\kappa$  induces a group isomorphism of  $\langle \tau_0 \rangle^{\mathbb{Z}/p^M}$  and  $\langle h_0 \rangle^{\mathbb{Z}/p^M}$ . Now Proposition 4.1 implies that  $\kappa_{< p}$  is isomorphism.  $\square$

Under the isomorphism  $\kappa_{< p}$ , the subfields  $\mathcal{K}[s, M] \subset \mathcal{K}_{< p}$ , where  $1 \leq s < p$  (cf. Subsection 3.5), give rise to the subfields  $K[s, M] \subset K_{< p}$  such that  $\text{Gal}(K[s, M]/K) = \Gamma/\Gamma^{p^M}C_{s+1}(\Gamma)$ . In other words, the extensions  $K[s, M]$  appear as the maximal  $p$ -extensions of  $K$  with the Galois group of period  $p^M$  and nilpotent class  $s$ .

Using that the identification  $\mathcal{G} = \tilde{\Gamma}$  is compatible with ramification filtrations, cf. Subsection 4.2, we obtain the following result about the maximal upper ramification numbers of the field extensions  $K[s, M]/K$ , where  $M \in \mathbb{N}$  and  $1 \leq s < p$ .

**Theorem 4.5.** *If  $[K : \mathbb{Q}_p] < \infty$ ,  $e_K$  is the ramification index of  $K$  and  $\zeta_M \in K$  then for  $1 \leq s < p$ ,*

$$v(K[s, M]/K) = e_K \left( M + \frac{s}{p-1} \right) - \frac{1 - \delta_{1s}}{p}.$$

*Proof.* Note first, that the Herbrand function  $\varphi_{\tilde{K}/K}(x)$  is continuous for all  $x \geq 0$ ,  $\varphi_{\tilde{K}/K}(0) = 0$  and its derivative  $\varphi'_{\tilde{K}/K}$  equals 1 if  $x \in (0, e^*)$  and equals  $p^{-m}$ , if  $m \in \mathbb{N}$  and  $x \in (e^*p^{m-1}, e^*p^m)$ .

From Proposition 3.8 we obtain that

$$v(K[s, M]/K) = \max \left\{ v(K(\pi_M)/K), \varphi_{\tilde{K}/K}(p^{M-1}(se^* - 1)) \right\}.$$

Note that  $v(K(\pi_M)/K) = \varphi_{\tilde{K}/K}(p^{M-1}e^*) = e^* + e_K(M-1)$  and, therefore,

$$v(K[1, M]/K) = v(K(\pi_M)/K) = e_K \left( M + \frac{1}{p-1} \right).$$

If  $2 \leq s < p$  then  $v(K[s, M/K])$  equals  $\varphi_{\tilde{K}/K}(p^{M-1}(se^* - 1)) =$   
 $= \varphi_{\tilde{K}/K}(p^{M-1}e^*) + \frac{p^{M-1}(se^* - 1) - p^{M-1}e^*}{p^M} = e_K \left( M + \frac{s}{p-1} \right) - \frac{1}{p}.$   
□

## REFERENCES

- [1] V.A.ABRASHKIN, *Ramification filtration of the Galois group of a local field*, Proceedings of the St. Petersburg Mathematical Society III, Amer. Math. Soc. Transl. Ser. 2, (1995) **166**, Amer. Math. Soc., Providence, RI
- [2] V.A. ABRASHKIN, *Ramification filtration of the Galois group of a local field. II*, Proceedings of Steklov Math. Inst. **208** (1995)
- [3] V.ABRASHKIN, *Ramification filtration of the Galois group of a local field. III*, Izvestiya RAN: Ser. Mat., **62**, no.5 (1998), 3-48; English transl. Izvestiya: Mathematics **62**, no.5, 857-900
- [4] V.ABRASHKIN *On a local analogue of the Grothendieck Conjecture* Int. J. Math., **11** (2000), no.1, 3-43
- [5] V.ABRASHKIN *Modified proof of a local analogue of the Grothendieck Conjecture* J.Théor. Nombres Bordeaux, **22** (2010), no.1, 1-50
- [6] V.ABRASHKIN, R.JENNI *The field-of-norms functor and the Hilbert symbol for higher local fields* J.Théor. Nombres Bordeaux, **24** (2012), no.1, 1-39
- [7] V.ABRASHKIN, *Galois groups of local fields, Lie algebras and ramification*, Arithmetic and Geometry, London Mathematical Society Lecture Note Series: **420**, Cambridge University Press, 2015, pp.1-23
- [8] V.ABRASHKIN, Automorphisms of local fields of period  $p$  and nilpotent class  $< p$ . (arXiv:1403.4121)
- [9] J.-M.FONTAINE, *Representations  $p$ -adiques des corps locaux (1-ere partie)*. In: The Grothendieck Festschrift, A Collection of Articles in Honor of the 60th Birthday of Alexander Grothendieck, vol. II, 1990, 249-309
- [10] M. HALL The theory of groups, The Macmillan Company New York, 1959
- [11] E.I. KHUKHRO,  *$p$ -automorphisms of finite  $p$ -groups*. London Mathematical Society Lecture Note Series, **246**. Cambridge University Press, Cambridge, 1998. xviii+204 pp.
- [12] M. LAZARD, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Ecole Norm. Sup. (1954) **71**, 101-190
- [13] SH. MOCHIZUKI, *A version of the Grothendieck conjecture for  $p$ -adic local fields*, Int. J. Math. (1997) **8**, no.4, 499-506
- [14] J.-P. SERRE Local fields, Berlin, New York: Springer-Verlag, 1959
- [15] J.-P. WINTENBERGER, *Le corps des normes de certaines extensions infinies des corps locaux; application*, Ann. Sci. Ecole Norm. Sup., IV Ser, (1983) **16**, 59-89

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